This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.
+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.
+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.
+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at http://books.google.com/

5
7
1
.

[redacted]

[redacted]

[redacted]

50
2
150
100
115.0

KURT F. WENDT LIBRARY
COLLEGE OF ENGINEERING
UNIVERSITY OF WISCONSIN
MADISON, WI 53706

A TEXT-BOOK

ON

HYDRAULICS

INCLUDING AN OUTLINE OF THE

THEORY OF TURBINES

BY

L. M. HOSKINS

Professor of Applied Mathematics in the
Leland Stanford Junior University

THIRD EDITION, REVISED

NEW YORK

HENRY HOLT AND COMPANY

1911

Copyright, 1906,
BY
HENRY HOLT AND COMPANY

6492869

PREFACE.

THIS book is designed primarily for the use of students of engineering in universities and technical colleges. In its preparation the aim has been to present fundamental principles in a manner both sound and as simple as possible. The treatment presupposes a good elementary knowledge of the principles of mechanics, and a working knowledge of the elements of calculus; but to the student thus equipped, who is also well trained in arithmetic, algebra and trigonometry, it presents little mathematical difficulty. Many numerical examples are introduced, the complete solution of which should form an important part of the work of the student.

It is perhaps not too much to say that the key to a correct understanding of all problems in the steady flow of liquids is supplied by Bernoulli's theorem,—or, as it is usually called in the text, the general equation of energy. Familiarity with this principle is therefore much more important than a memory-knowledge of special rules, and for this reason the explanations of particular cases of flow have in most cases been based directly upon the fundamental equation. The meaning and importance of the term representing lost energy in this equation have also been emphasized. The corresponding theory applied to gases is given in Appendix A.

In the presentation of working rules for estimating flow in the various practical cases met by the engineer it has been aimed in every case to give a clear statement of the rational basis of the formula adopted, and also to make clear to what extent the theory is defective and the formula therefore empirical. It has also been attempted to avoid the appearance of

precise knowledge where the reality is absent. For example, no elaborate tables have been given purporting to show accurately how frictional loss of head in pipes depends upon velocity and diameter, or giving precise values of the friction factor for pipes of different kinds. A somewhat careful study of experimental data has failed to convince the author of the reliability of any such tables.

The treatment of turbines and water wheels has been restricted to an outline of the theory, but several illustrations showing typical American practice have been included. For these the author is indebted to the courtesy of manufacturers, to whom credit is in every case given in the text. The aim has been to unify the theory, the treatment of all specific cases being based upon the same general principles and equations, and a general notation for velocities and their direction-angles being adopted which it is hoped will be found simple and helpful. This unification includes the theory of turbine pumps.

In various discussions throughout the book reference is made to the author's text-book on Theoretical Mechanics for a fuller explanation of basal principles.

L. M. H.

PALO ALTO, CAL., June, 1906.

PREFACE TO THIRD EDITION.

IN this edition the only important changes and additions are in Chapter XIII and in Appendix A. In the former the treatment of weir formulas has been partly rewritten, while to the latter the ordinary theory of flow of a gas in a uniform pipe has been added. It is hoped that few if any typographical or other errors remain uncorrected.

PALO ALTO, CAL., February, 1911.

CONTENTS.

HYDRAULICS

CHAPTER I.

PRELIMINARY DEFINITIONS AND PRINCIPLES.

1. Definition of Subject.—The mechanics of fluid bodies is called *Hydromechanics*. It embraces *Hydrostatics*, dealing with the principles of fluid equilibrium, and *Hydrokinetics*, dealing with the laws of fluid motion.

Hydraulics, the subject of this book, may be defined briefly as practical Hydromechanics. It deals especially with the flow of water in streams of various kinds, but may be taken to include all the principles and applications of Hydromechanics that bear directly upon problems of practical utility. Many of the laws of Hydraulics are largely empirical, but certain fundamental dynamical principles, especially the law of energy, serve to unify the subject and to put it upon a scientific basis.

The bodies dealt with in Hydromechanics may be either liquids or gases. Hydraulics deals mainly, but not exclusively, with liquids, and especially with water.

2. Distinction between Solid and Fluid Bodies.—A solid body can permanently resist change of shape; a fluid body cannot.

A fluid is either liquid or gaseous. A gas tends to expand indefinitely, so as to fill any continuous closed volume in any portion of which it may be placed. A liquid changes its volume only slightly under changes of pressure; a given portion may be wholly freed from external pressure without expanding beyond a certain volume.

The distinction between solids and fluids may be made more definite by a consideration of internal stresses.

3. Internal Stresses in a Body.—The two equal and opposite forces exerted by two portions of matter upon each other constitute a stress. The forces making up a stress are thus the "action and reaction" of Newton's third law of motion.

If the two portions of matter are parts of the same body, the stress is *internal* with reference to that body. Internal stresses, acting between adjacent portions of a body, are called into action whenever external forces tend to change the shape or size of the body.

If a plane surface be conceived to divide a body into two contiguous parts, the forces which these parts exert upon each other will for convenience be regarded as resolved into components normal and tangential to the plane. The stress composed of these forces is thus resolved into

(*a*) A normal stress, which resists whatever tendency there may be for the two parts of the body to approach or recede from each other in the direction of the normal to the plane of separation, and

(*b*) A tangential stress, which resists any tendency to sliding, or relative motion parallel to the plane.

4. Mathematical Definitions of Solid and Fluid.—A fluid body is one in which tangential stress cannot act, except while the shape of the body is changing.

If, by reason of external forces, any two adjacent portions of a fluid tend to slide over each other, tangential stresses come into action to resist such sliding. The sliding is not prevented, however, but continues until a condition of equilibrium is attained; in this condition the tangential stress on every plane vanishes.

A solid body is one in which tangential stresses can act permanently to resist change of shape.

A "perfect" fluid may be defined as one which offers no resistance to change of shape. In other words, no tangential

stress acts in a perfect fluid even while the particles are sliding over one another. No known fluid is perfect in this sense.

The laws of equilibrium are the same for an actual fluid as for a perfect fluid, since it is only when the parts of a body of fluid move relatively to one another that tangential stresses act. In other words, the statics of actual fluids is the same as the statics of perfect fluids.

5. Pressure; Intensity of Pressure.—The normal stress between two adjacent parts of a body may be either tensile or compressive.

Tensile stress (or tension) resists a tendency of the two portions of the body to separate.

Compressive stress (or pressure) resists a tendency of the two portions of the body to approach each other.

In Hydromechanics we are concerned mainly with pressure, since a fluid body can sustain only a slight tensile stress. We shall have to consider not only *internal* pressure, but also pressure acting between a body of water and other bodies in contact with it.

Intensity of pressure means pressure per unit area. If, on any surface subject to pressure, the pressures upon any two elementary areas, however small, are proportional to the areas, the intensity of pressure is uniform over the surface. In this case its value is at every point equal to the total pressure divided by the total area. Algebraically, let

P = total pressure on area F;

p = intensity of pressure at any point of the area;

then
$$p=\frac{P}{F}.$$

If the intensity of pressure has not the same value at all points of the area, we may regard P/F as its average value for the area F. The true value of p at any given point may be expressed approximately as its average value over a small area containing the point. If ΔF is the area of this element

and ΔP the total pressure on the element, we have approximately

$$p = \frac{\Delta P}{\Delta F},$$

the approximation being closer the smaller the element.

Taking ΔF smaller and smaller with limit 0, but always containing the point at which the value of p is to be expressed, we have as the exact value of the intensity of pressure at that point

$$p = \text{limit } \frac{\Delta P}{\Delta F} = \frac{dP}{dF}.$$

For brevity the word "pressure" is often used instead of "intensity of pressure." This abbreviation will sometimes be employed in the following discussions, but should be avoided when it is liable to cause ambiguity.

Although the foregoing discussion of intensity of stress has referred to normal stresses only, similar considerations hold for tangential stresses.

6. Elasticity.—Elasticity is the property by virtue of which a body regains its original size and shape (in whole or in part) after these have been changed by the action of external forces.

Both elasticity of volume and elasticity of shape are possessed in very different degrees by different bodies. Fluid bodies possess practically perfect elasticity of volume, but no elasticity of shape.

Since change of shape always involves sliding, or tangential motion of the parts of a body relative to one another, the body cannot of itself regain its original shape except by the continued action of the tangential stresses which resist such motion. A fluid body, having been deformed from one form of equilibrium to another, cannot of itself return to the original shape, because after equilibrium is attained no tangential stresses are in action.

7. External Forces.—A force acting upon any portion of a body is called external if the portion of matter exerting the force is not a part of the body.

The external forces acting upon any body of fluid are of two classes: (*a*) surface forces and (*b*) bodily forces.

(*a*) A *surface force* is one whose place of application is some portion of the bounding surface of the body. Such forces are exerted by other bodies in contact with the given body.

(*b*) A *bodily force* is one which is applied throughout some definite volume of the body. Such a force does not depend upon contact between the two bodies concerned; it is of the kind called "action at a distance." An example of a bodily force is gravity, acting upon every particle of a body of fluid near the earth's surface. In practical Hydraulics this is the only bodily force to be considered.

8. Density and Compressibility of Water. — Practical Hydraulics deals mainly with water. The properties of water with which we shall chiefly be concerned are its density and compressibility.

Density.—The density (or mass per unit volume) of water varies but little with pressure and temperature within ordinary ranges. The density of pure water at several different temperatures and under ordinary atmospheric pressure is given in the following table:

TABLE I.

Temp. Fahr.	Density in lbs. per cu. ft.	Temp. Cent.	Density in kgr. per cu. met.
32°	62.42	0°	999.9
39.3	62.424	4	1000.0
50	62.41	10	999.8
60	62.37	15	999.2
70	62.30	20	998.3
80	62.22	25	997.1
90	62.12	30	995.8
100	62.00	35	994.6
110	61.86	40	992.4

The density of ordinary terrestrial water is increased very slightly by the presence of various substances in solution. In

most ordinary computations the convenient numbers 62.5 lbs. per cubic foot and 1000 kilograms per cubic meter may be employed; 62.4 lbs. per cubic foot is, however, more accurate than the former value.

The density of sea-water is about 2.6 per cent greater than that of pure water.

Compressibility.—The compressibility of water is so small that for ordinary purposes it may be neglected. The ratio to the original volume of the decrease in volume caused by a given pressure (of uniform intensity over the whole bounding surface) may be taken as a measure of the compressibility. The value of this ratio for a pressure of 1 atmosphere * varies somewhat with the temperature. Near the freezing-point it is about 0.000050; at 77° Fahr. (25° Cent.) it is about 0.000045.

* See Art. 14.

CHAPTER II.

HYDROSTATICS.

9. Pressure on Different Planes Passing Through a Point.— From the fact that no tangential stress exists on any plane in a body of fluid in equilibrium, it follows that the intensity of normal pressure has, at a given point, the same value for all planes passing through that point.

Consider any two planes passing through the point O (Fig. 1). Let OM, ON be their traces on a plane perpendicular to both which is taken as the plane of the figure. Take $OA = OB$, and let AB be the trace of a third plane which is perpendicular to the plane of the figure. Consider the body of fluid bounded by five planes as follows: The plane of the figure; a plane parallel to it at a distance z; planes perpendicular to the plane of the figure having traces OA, OB, AB. The forces acting upon this body are the normal pressures exerted upon every part of its bounding surface by the surrounding fluid, and whatever bodily forces may be acting upon it.

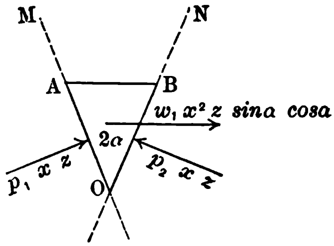


FIG. 1.

Let the average value of the bodily force per unit volume, for the body just described, be w', and let

w_1 = component of w' in direction AB;

p_1 = average intensity of pressure on the face OA;

p_2 = average intensity of pressure on the face OB;

2α = angle AOB;

$x = OA = OB$.

Then $p_1xz =$ total pressure acting on face OA;

$p_2xz =$ " " " " " OB;

$w_1x^2z \sin\alpha \cos\alpha =$ total bodily force in direction AB.

These are the only forces acting on the body which are not perpendicular to AB; hence for equilibrium, resolving in direction AB,

$$p_1xz \cos\alpha - p_2xz \cos\alpha + w_1x^2z \sin\alpha \cos\alpha = 0,$$

or

$$p_1 - p_2 + w_1x \sin\alpha = 0.$$

This equation is true for any values of x and z. If both be made to approach 0, the third term of the equation approaches 0, so that

$$\text{limit } p_1 = \text{limit } p_2.$$

But the limiting values of p_1 and p_2 are the true values of the intensity of pressure at the point O on the planes OA and OB respectively. And since these may be any two planes passing through O without changing the reasoning, the proposition is established.

10. Normal Pressure in a Fluid Free from Bodily Forces.—Although in all the practical problems of Hydraulics and Hydrostatics the fluids considered are acted upon by the force of gravity, it is instructive to consider the ideal case of a fluid not acted upon by any bodily force.

It is easily shown that for such a body in equilibrium the intensity of pressure has the same value at all points.

Let A and B (Fig. 2) be any two points such that the straight line AB lies wholly within the fluid. Consider an elementary prism of small cross-section F, so taken that A and B lie in its two bases. Let p_1 and p_2 denote the values of the intensity of pressure at A and B respectively, and let all forces acting upon the prismatic element be resolved parallel to its axis. The only forces not normal to the direction of resolution are the pressures on the end elements at A and B; hence for equilibrium we have

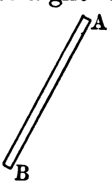


FIG. 2.

$$p_1F - p_2F = 0, \quad \text{or} \quad p_1 = p_2.$$

That is, the intensity of pressure has the same value at A and at B. This result may obviously be extended to any two points in a continuous body of fluid free from the action of bodily forces.

11. Variation of Pressure in Fluid Acted upon by Gravity.— In case of fluids at the earth's surface the only bodily force to be considered is the attraction of the earth upon every particle. For a fluid of uniform density this attraction (per unit volume) has practically the same magnitude and direction at all points. If the weight of a pound mass (called a pound force) is taken as the unit force, the force per unit volume has the same value as the mass (in pounds) per unit volume.

The law of variation of pressure in a body of fluid acted upon by gravity may be determined as follows:

Let A and B (Fig. 3) be any two points such that the straight line AB lies wholly in the fluid. Let BC be horizontal and AC vertical, and let $AB = l$, $AC = z$.

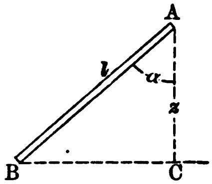


FIG. 3.

Consider an elementary right prism with axis AB and cross-section F, the bases containing the points A and B. Let all forces acting upon this prism be resolved parallel to AB. Let

p_1 = intensity of pressure at A;
p_2 = intensity of pressure at B;
w = weight of fluid per unit volume;
α = angle BAC.

For equilibrium,

$$p_1F - p_2F + wFl \cos \alpha = 0.$$

Or, since $l \cos \alpha = z$,

$$p_2 - p_1 = wz.$$

This result may be extended to any two points of a connected fluid, and may be stated in general terms as follows:

In any body of homogeneous fluid in equilibrium the pres-

sure varies in direct ratio with the depth. The difference between the values of the pressure (per unit area) at any two points is equal to the weight of a prism of the fluid of unit cross-section and of length equal to the vertical distance between the two points.

The following discussions will refer usually to the case of a liquid acted upon by gravity.

Units.—It is to be remembered that a consistent system of units must be used in applying the above formula. Throughout this book the foot will nearly always be used as the unit length, and the pound as the unit force. The value of w is therefore 62.5 (more accurately 62.4) pounds per cubic foot, z is to be expressed in feet, and p in pounds per square foot. With French units, if the meter is taken as the unit length and the kilogram as the unit force, the value of w is the weight in kilograms of a cubic meter of water, or 1,000.

12. Surface of Equal Pressure.—As a special case of the above proposition it is seen that if the intensity of pressure has the same value at any two points, these must lie in the same horizontal plane. Any horizontal plane is, in fact, a *surface of equal pressure.*

A surface of equal pressure is called a *level surface.*

Free surface.—The bounding surface of a liquid is said to be free if not sustaining pressure. Ordinarily the upper surface of a body of water is called free although under atmospheric pressure. Whether under zero pressure or any uniform pressure, the free surface of a liquid in equilibrium is obviously a horizontal plane * if gravity is the only bodily force.

13. Pressure Expressed in Terms of Height of Liquid Column.—Fluid pressure is often estimated in terms of the "equivalent height" of some specified liquid. Thus, a column of water 1 foot high is said to be "equivalent to" a pressure of about

* The accurate statement is that the free surface (or any level surface) is everywhere normal to the direction of gravity, and is approximately a spherical surface concentric with the earth.

62.5 pounds per square foot, since the intensity of pressure at the base of such a column exceeds that at the top by the amount stated. In discussions in Hydraulics it is quite common to express pressures in this way.

In scientific investigations mercury is often taken as the standard fluid, pressures being expressed as so many inches, or centimeters, of mercury.

Since the density of mercury is about 13.6 times that of water, the height of the water column equivalent to a given pressure is about 13.6 times as great as that of the corresponding mercury column.

14. Atmospheric Pressure.—The air exerts upon all terrestrial bodies a pressure whose intensity is equal to the weight of a column of air of unit cross-section extending upward completely through the atmosphere. In many hydraulic problems this pressure may be disregarded because its effects at different points counterbalance. It is quite common to reckon pressures from atmospheric pressure as zero, pressures of less intensity being regarded as negative. An actual negative pressure (i.e., a tensile stress) of any considerable intensity cannot exist in a liquid, any tendency to such a stress resulting in a separation of the parts of the liquid.

The intensity of atmospheric pressure at any locality varies somewhat with weather conditions, and at different places it varies with the elevation. At sea-level under ordinary conditions the value is about 14.72 pounds per square inch, or 2120 pounds per square foot.

The height of the equivalent water column is very nearly 34 feet (10.34 meters), and that of the equivalent mercury column 30 inches (76 centimeters).

At any height above sea-level the corresponding values may be found by multiplying the above numbers by the proper factor taken from the following table, which gives the ratio of atmospheric pressure at any elevation to its value at sea-level. It must be understood that the results will be only approximate, the table being computed for the

ideal case of a perfect gas at the uniform temperature of 0° Cent.

TABLE II.

Elevation above sea-level in feet.	Ratio of atmospheric pressure to its value at sea-level.	Elevation above sea-level in feet.	Ratio of atmospheric pressure to its value at sea-level.
500	.9811	8,000	.7370
1,000	.9626	8,500	.7231
1,500	.9444	9,000	.7094
2,000	.9265	9,500	.6960
2,500	.9090	10,000	.6829
3,000	.8918	10,500	.6699
3,500	.8750	11,000	.6573
4,000	.8584	11,500	.6449
4,500	.8422	12,000	.6327
5,000	.8263	12,500	.6207
5,500	.8107	13,000	.6090
6,000	.7954	13,500	.5975
6,500	.7804	14,000	.5862
7,000	.7656	14,500	.5751
7,500	.7512	15,000	.5642

15. Resultant Pressure.—The resultant pressure on any surface, plane or curved, is found by combining the pressures on the elementary portions of the surface, having regard for direction as well as magnitude.

The point of application of the resultant pressure, for a given surface, is called the *center of pressure.*

16. Resultant Pressure on Horizontal Plane Area. — Let

F = area of a horizontal plane surface;
z = its depth below free surface of liquid;
p = intensity of pressure at any point;
P = resultant pressure on area F.

Then
$$p = wz;$$
$$P = Fp = wzF.$$

The center of pressure (or point of application of P) is evidently coincident with the centroid of the area; for the forces

of which P is the resultant are parallel, and are proportional to the elementary areas on which they act.

17. Magnitude of Resultant Pressure on any Plane Area.—If a submerged plane area is not horizontal, the magnitude of the resultant pressure may be computed as follows:

Let AB and $A'B'$ (Fig. 4) represent two vertical projections of the surface, its inclination to the horizontal being θ. Let

z = depth of an elementary area dF below free surface of liquid;

P = resultant pressure on whole area F.

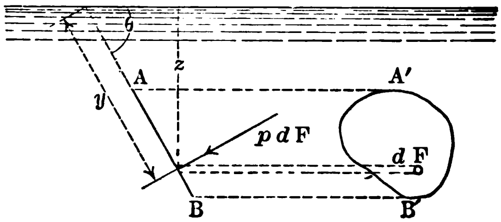


FIG. 4.

Then the total pressure on the element dF is

$$dP = p\,dF = wz\,dF;$$

hence
$$P = \int p\,dF = w\int z\,dF,$$

the integration being extended over the whole area F.

If $\bar{z}$ is the value of z at the centroid of the area F, we have

$$\int z\,dF = \bar{z}F; \quad \text{therefore} \quad P = w\bar{z}F.$$

18. Center of Pressure of Plane Area.—To determine the point of application of the resultant pressure P, the principle of moments must be employed.

Referring to Fig. 4, let the line of intersection of the plane AB with the free surface be taken as axis of moments, and let the moment of the resultant pressure be equated to

the sum of the moments of the pressures on all elementary areas.

Let the distance of an elementary area dF from the axis of moments be denoted by y, the corresponding vertical distance being z, so that $z=y \sin \theta$. Let $\bar{z}$, $\bar{y}$ refer to the centroid of the area F, and z', y' to the center of pressure.

The equation of moments is

$$Py'=\int y\, dP,$$

the integration covering the whole area F.

Since $dP=wz\, dF=wy \sin \theta \cdot dF$, and $P=w\bar{z}F=w\bar{y} \sin \theta \cdot F$, the equation may be written

$$wF\,\bar{y}y' \sin \theta=w \sin \theta \cdot \int y^2\, dF;$$

from which

$$y'=\frac{\int y^2\, dF}{\bar{y}F}.$$

The numerator of this value is equal to the moment of inertia of the area with respect to the line in which its plane intersects the free surface of the water; the denominator is the statical moment of the area with respect to the same axis. Denoting these quantities by I and G respectively, we have

$$y'=\frac{I}{G}. \quad \ldots \ldots \ldots \quad (1)$$

If the area has an axis of symmetry which is perpendicular to the line in which its plane intersects the water surface, the center of pressure lies in this axis and is completely determined by equation (1). If this is not the case, a second moment-equation is required for the complete location of the center of pressure.

Taking as axis of moments a line lying in the area and perpendicular to the axis from which y is measured, let x denote

the distance from this axis of an element dF; then the equation of moments is

$$Px' = \int x\,dP,$$

which may be written

$$wF\bar{y}x' \sin\theta = w \sin\theta \cdot \int xy\,dF;$$

from which

$$\therefore x' = \frac{\int xy\,dF}{\bar{y}F} = \frac{J}{G}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Here J denotes the product of inertia of the area F with respect to the axes from which x and y are measured.

EXAMPLES.

Find the magnitude and point of application of the resultant pressure on the area described in each of the following examples, the liquid being water.

1. A rectangle of sides 4 ft. and 6 ft., placed vertically, (a) with shorter side in water surface, (b) with longer side in water surface.
Ans. (a) $P = 4500$ lbs.; $y' = 4$ ft.

2. A rectangle with sides b, d, placed vertically, with the side b in the water surface. *Ans.* $P = wbd^2/2$; $y' = \frac{2}{3}d$.

3. A circle of diameter d, its plane being inclined at angle θ to the vertical, and the center being distant a vertically below the surface.
Ans. $P = w\pi d^2 a/4$; $y' = a \sec\theta + d^2/16a \sec\theta$.

4. A circle of 2 ft. diameter, the highest point being 6 inches below the surface, and the plane inclined 60° to the vertical.
Ans. $P = 196$ lbs.; $y' = 2\frac{1}{8}$ ft.

5. A semicircle with plane vertical and diameter in the surface.

6. A triangle of base 2 ft. and altitude 3 ft., the plane being vertical, (a) with vertex in water surface and base horizontal, (b) with base in surface. *Ans.* (a) $y' = \frac{3}{4}d$. (b) $y' = \frac{1}{2}d$.

7. An area F, whose radius of gyration about a horizontal central axis is k, placed vertically, with centroid at depth a below water surface.
Ans. $P = wFa$; $y' = a + k^2/a$.

8. The vertical plane area shown in Fig. 5.

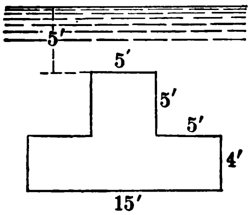


FIG. 5.

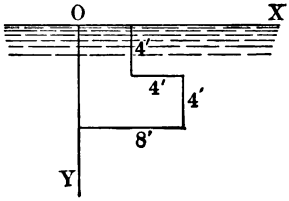


FIG. 6.

9. The vertical plane area shown in Fig. 6.

Ans. $P=14{,}000$ lbs.; $x'=3.71$ ft., $y'=5.71$ ft.

19. Pressure Resolved in Given Direction.—The component, in any direction, of the resultant pressure on a plane area may be found by the following rule:

Pass a plane through the centroid of the area perpendicular to the given direction, and project the area orthographically upon it; the pressure on this projected area is equal to the required component of the resultant pressure on the given area.

Thus, let it be required to find the component, in the direction MN, of the resultant pressure on the area AB (Fig. 7).

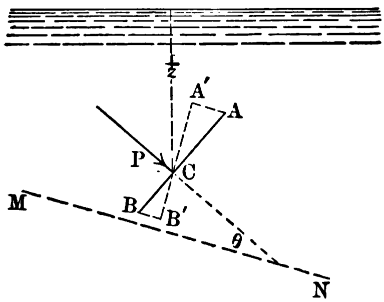


FIG. 7.

The value of the resultant pressure is $P=wF\bar{z}$, $\bar{z}$ being the depth below the free surface of C, the centroid of the area F. Through C pass a plane perpendicular to MN, and let $A'B'$ be the projection of AB on this plane. The projected area is $F'=F\cos\theta$, where θ is the angle between P and MN; the

centroid of the projected area coincides with that of the given area AB. Resolving P in the direction MN, we have

$$P \cos \theta = wF \bar{z} \cos \theta,$$

while the resultant pressure on $A'B'$ is

$$wF'\bar{z} = wF \bar{z} \cos \theta.$$

These values being equal, the proposition is proved.

EXAMPLES.

1. Compute the horizontal and vertical components of the resultant pressure on a rectangular area 6 ft. by 8 ft., inclined 30° to the vertical, one of the longer edges being in the water surface.

Ans. Hor. comp.=6750 lbs.; vert. comp.=3900 lbs.

2. Compute the horizontal and vertical components of the resultant pressure on a circular area 4 ft. in diameter, inclined 20° to the vertical, the center being 5 ft. below the water surface.

Ans. Hor. comp.=3690 lbs.; vert. comp.=1345 lbs.

20. Horizontal Pressure on Curved Surface.—In case the surface is not plane, the resolved pressure in a given direction cannot in general be computed by the rule above given for a plane surface (Art. 19). The rule does hold, however, if the direction of resolution is horizontal.

Thus, consider the pressure upon the surface AB of the submerged body X (Fig. 8). This pressure is identical with

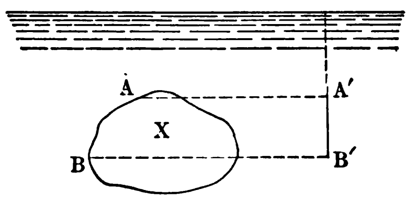


FIG. 8.

that upon the surface AB of the liquid which would replace X if removed.

Let $A'B'$ be the orthographic projection of the surface AB upon any vertical plane, and consider the body of fluid $ABB'A'$. The horizontal component of the pressure on AB is counterbalanced by the pressure on $A'B'$, and must therefore have the same magnitude and line of action.

EXAMPLES.

1. A circular cylinder 4 ft. long and 2 ft. in diameter is placed with axis horizontal, and is filled with water to a depth of 18 inches. Compute the magnitude and line of action of the horizontal thrust of the water in a direction perpendicular to the axis of the cylinder.

Ans. 281 lbs. acting in line 1 ft. below water surface.

2. A hemispherical bowl 2 ft. in diameter is filled with water. Determine the magnitude and line of action of the resultant thrust of the water on the bowl in a given horizontal direction.

Ans. 41.7 lbs.; .59 ft. below surface.

3. A cylindrical barrel 2 ft. in diameter is filled with water to a depth of 30 inches. Determine the magnitude and line of action of the resultant horizontal thrust on half the interior surface

Ans. 391 lbs.; 20 inches below water surface.

21. Vertical Pressure on Curved Surface.—Let AB (Fig. 9) be a portion of the surface of a submerged body X, and let it be required to compute the resultant of the vertical components

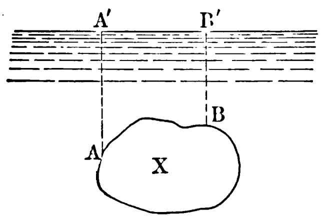


Fig. 9.

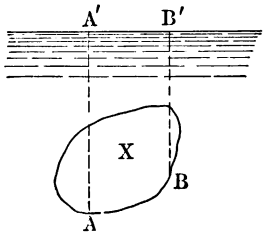


Fig. 10.

of the pressures exerted by the water upon the elements of AB. Let $A'B'$ be the orthographic projection of AB upon the plane of the water surface. The volume of water $ABB'A'$ is in equilibrium under the action of its weight and the pres-

sures acting on its bounding surface. The surface $A'B'$ being assumed free from pressure,* the resolution of these forces vertically shows that the vertical component of the pressure on AB is equal to the weight of the body of water $ABB'A'$, and that its line of action passes through the center of gravity of this body.

In the case shown in Fig. 9, the resultant vertical pressure of the water on the surface AB of the body X acts downward; while in such a case as that shown in Fig. 10 it acts upward. But in the latter case, as in the former, the vertical pressure is equal in magnitude to the weight of a body of water $ABB'A'$.

22. Resultant Pressure on Curved Surface.—The resultant pressure on a curved surface is not, in the most general case, a single force, since a system of forces in three dimensions is not generally equivalent to any single force. In practical problems it will usually suffice to determine the effective pressures in horizontal and vertical directions, by the methods just explained. If it is required to carry the reduction of the system farther, this may be done by methods explained in works on Statics.† It is sometimes evident from symmetry that the total pressure is equivalent to a single force. This is true in the following case.

23. Pressure on Masonry Dam.—In discussing the stability of a dam it is necessary to compute the resultant pressure exerted upon it for a given length, say one foot. The horizontal and vertical components of this pressure may be computed by the rules above given, and since these components act in the same vertical plane, they have a single resultant which may readily be determined.

* Atmospheric pressure is usually neglected, because in practical cases it is commonly the excess of pressure above that due to the atmosphere that is of importance. It may, however, be taken account of in the present case by conceiving the column of water $ABB'A'$ extended to a height p_0/w above the actual water surface, p_0 being atmospheric pressure.

† Theoretical Mechanics, Chapter X.

In Fig. 11, the total horizontal pressure H on the face AB is equivalent to the resultant pressure upon $A'B$, the projection of AB on a vertical plane; while the vertical pressure V is equal to the weight of the body of water $AA'B$. The line of action of H passes through the center of pressure of $A'B$, while that of V passes through the center of gravity of $AA'B$; their intersection gives a point in the line of action of P, the resultant of H and V.

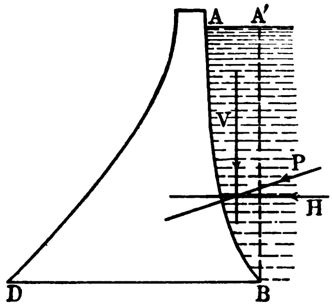


FIG. 11.

In masonry dams as actually constructed, the profile AB often differs but little from a vertical straight line, and the vertical component of the pressure is neglected in discussing the stability of the dam. This is an error on the side of safety in the design.

Upward pressure on base of dam.—If a masonry dam rests wholly or partly upon a bed of porous material such as sand or gravel, which is continuous with the bed of the reservoir so that water freely enters and saturates it, an upward pressure results which may have an important effect upon the stability of the dam. If the water is at rest throughout the porous bed, the upward pressure must be computed as due to the column of water $A'B$ (Fig. 11). If the water flows through the gravel, as it will unless intercepted by an impervious barrier, the pressure will decrease in the direction of flow (Arts. 90–93).

Even if the dam rests upon impervious rock, a horizontal fissure in the masonry may permit the entrance of water, thus causing an upward pressure upon the masonry above.

The following examples refer to a dam of the cross-section shown in Fig. 11, the dimensions being as given below. In all cases the required pressures are to be computed for one linear foot of the dam.

Referring to the figure, let y denote depth below water surface, x the corresponding horizontal distance from $A'B$ to AB,

and x' the horizontal thickness of the masonry. The values given in the table are in feet.

y	x	x'	y	x	x'
−10	34.8	17.5	80	29.8	57.5
0	34.3	19.2	90	28.5	66.0
10	33.8	21.7	100	26.6	75.7
20	33.4	24.6	110	24.2	86.5
30	32.9	28.5	120	21.2	98.2
40	32.4	33.0	130	17.6	111.0
50	31.9	38.2	140	12.9	125.3
60	31.5	44.0	150	7.2	140.7
70	30.6	50.4	160	0.0	159.5

EXAMPLES.

1. Determine the magnitude, direction, and line of action of the resultant pressure on the surface AB.

2. Assuming the dam to rest upon saturated gravel for its entire thickness BD, compute the total upward force if the pressure is everywhere due to the head $A'B$.

3. Compute the total upward force if the pressure head varies uniformly from 0 at D to $A'B$ at B.

4. If the material of the dam weighs 150 lbs. per cu. ft., compute the magnitude and line of action of the weight per linear foot. Determine the magnitude, direction, and line of action of the resultant of the water pressure and the weight of the dam; also the moment of this resultant about D. (Solve on each of the above assumptions as to the pressure on the base.)

24. Curved Surface under Pressure of Uniform Intensity.— If a body is submerged to a depth which is great in comparison with the vertical dimension of the body, the variation of the intensity of pressure on its bounding surface will be small in comparison with its actual value, and for practical purposes this variation may usually be disregarded.

If a surface is under pressure of uniform intensity, the total resolved pressure in *any* direction is equal to the resultant pressure on the orthographic projection of the surface upon a plane perpendicular to that direction.

Thus, consider the pressure on the part AB of the surface of the body X (Fig. 12). If it is required to compute the resolved part of this pressure in any given direction, let $A'B'$ be the orthographic projection of AB upon a plane perpendicular to that direction. If X were removed, the body of water $ABB'A'$ would be in equilibrium under the action of the pressures exerted by the surrounding fluid (the weight of this body being by supposition negligible in comparison with these pressures). Resolving perpendicularly to $A'B'$, the resolved pressure on AB exactly balances the pressure on $A'B'$.

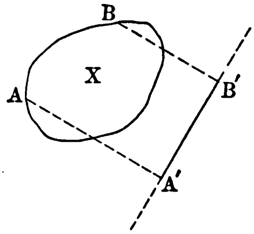


FIG. 12.

EXAMPLES.

1. A 6-inch pipe carries water under a head of 200 ft. Compute the total pressure per foot of length on one half the interior surface. *Ans.* 6250 lbs.

2. Compute the resultant pressure on a hemispherical surface 2 ft. in diameter subjected to a pressure of 5 atmospheres. *Ans.* 33,300 lbs.

25. Resultant Pressure on Submerged Body.—The resultant pressure exerted by a fluid upon a body which is wholly or partly submerged in it is a force equal and opposite to the weight of the displaced fluid, and its line of action passes through the center of gravity of the displaced fluid.

It is obvious that, if the body were removed, the body of fluid which would replace it would be subjected to exactly the same pressure as that actually exerted upon the submerged body. But the body of fluid would be in equilibrium under the action of two sets of forces: (*a*) the weight of every particle, with resultant acting through the center of gravity; (*b*) the pressure of the surrounding fluid upon every element of the bounding surface. The resultants of these two sets of forces must be equal in magnitude, opposite in direction, and collinear.

The resultant pressure of a liquid upon a body wholly or

partly submerged in it is called the "buoyant force." The centroid of the submerged volume (the point of application of the buoyant force) is the "center of buoyancy."

If a body is submerged partly in one fluid and partly in another, the resultant pressure exerted upon it by both is equal to the total weight of both fluids displaced. Thus, a body floating at the surface of water displaces a certain body of water and a certain portion of air. The weight of the displaced air is so small in most ordinary problems that it is often disregarded. In the following discussions relating to floating bodies the pressure of the air will not be considered unless specifically mentioned.

26. General Conditions of Equilibrium of a Floating Body.— If the only forces acting upon a floating body are its weight and the pressure cf the liquid, these two forces (or strictly sets of forces) must balance each other. This requires

(1) That the weight of the displaced water shall equal the weight of the body, and

(2) That the center of gravity of the body and that of the displaced water shall lie in the same vertical line.

From these two conditions it is possible to determine what volume of water will be displaced by a body of known weight; and in the case of bodies of regular shape to determine by inspection some or all of the possible positions of equilibrium.

EXAMPLES.

1. A plank 2″ by 12″ by 16′, weighing 40 lbs. per cu. ft., can carry what weight without sinking? *Ans.* 60 lbs.

2. A box closed on all sides is made of lumber 1″ thick weighing 45 lbs. per cu. ft. Its outside dimensions are 2′ by 3′ by 6′. If half filled with water, at what depth will it float?

3. A cone of specific gravity 0.5 floats with axis vertical and apex downward. The altitude being h and radius of base a, compute the depth of flotation.

4. Compute the depth of flotation of the cone described in Ex. 3 if floating with vertex upward.

27. Floating Body Acted Upon by Any Forces.—If a floating body is in equilibrium under the action of forces additional to the pressure of the liquid and the weight of the body, such additional forces must be included in applying the conditions of equilibrium.

Since the weight of the body and the buoyant force are both vertical, it is evident that, for equilibrium, the additional forces must be equivalent either to a vertical force or to a couple.

EXAMPLES.

1. A homogeneous rectangular parallelopiped 1′ by 2′ by 3′, of specific gravity 0.25, is half submerged in water, one face being horizontal. What force, besides its weight and the pressure of the water, must be acting to hold it in equilibrium?

2. If the same body is in equilibrium with a diagonal plane in the plane of the water surface, determine the magnitude, direction, and line of action of the resultant force acting upon the body in addition to its weight and the buoyant force.

3. A right prism whose bases are equilateral triangles floats in such a position that a median plane coincides with the plane of the water surface. If the specific gravity is 0.8, what force, besides gravity and the buoyant force, must be acting upon the body?

28. Stability of Equilibrium.—The equilibrium of a floating body is *stable, unstable,* or *neutral,* according as the body tends, after being slightly displaced, to return to the original position, to depart farther from it, or to remain in the new position.

The brief discussion of stability which follows is limited to the case in which the only forces acting on the body are its weight and the pressure of the liquid.

A complete discussion of stability would require a consideration of all possible displacements. These displacements may be resolved into translations and rotations.

For translations the nature of the equilibrium is obvious, being stable for vertical displacements and neutral for horizontal displacements. For rotations the nature of the equilibrium cannot be determined so simply.

29. Metacenter.—Let a floating body be displaced in such a way that the submerged volume remains constant. Let the body in the new position be represented by KMN (Fig. 13),

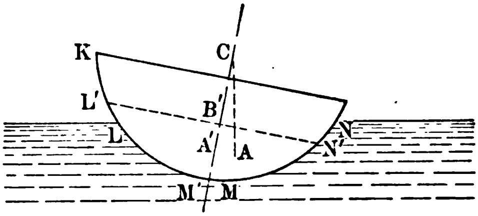


FIG. 13.

LN being the water surface; and let $L'N'$ be that plane in the body which coincided with the water surface in the original position of equilibrium.

Let A' be the centroid of the volume $L'M'N'$ (being therefore the position, in the body, of the center of buoyancy in the position of equilibrium), and A the centroid of the volume LMN (being therefore the center of buoyancy in the displaced position). Let B' be the center of gravity of the floating body.

In the original position the line $A'B'$ is vertical. In the new position let a vertical line through A intersect $A'B'$ (produced if necessary) in C. If the angle of displacement be taken smaller and smaller with zero as limit, C approaches a definite limiting position. This limiting position is called the "metacenter."

The stability of the equilibrium can be tested by determining the position of the metacenter.

It is seen that in the displaced position the forces acting upon the body are equivalent to the following couple: the weight of the body acting downward through B', and the buoyant force acting upward through A. If the metacenter C falls above B', the couple tends to bring the body back to the original position of equilibrium; if C falls below B', the couple tends to displace the body still farther. Hence in the former case the equilibrium is stable and in the latter unstable; while if the metacenter coincides with the center of gravity of the body the equilibrium is neutral.

A simple rule may be deduced for determining the position of the metacenter, and thus testing the stability of the equilibrium. In studying the stability of ships, however, it is not enough to test whether the equilibrium is stable for small displacements, but the degree of stability for both small and large displacements must be determined. The moment of the couple consisting of the weight of the body and the buoyant force, for any displacement, is a measure of the degree of stability. For a vessel of known shape and weight, the value of this moment may be computed for any angular displacement.

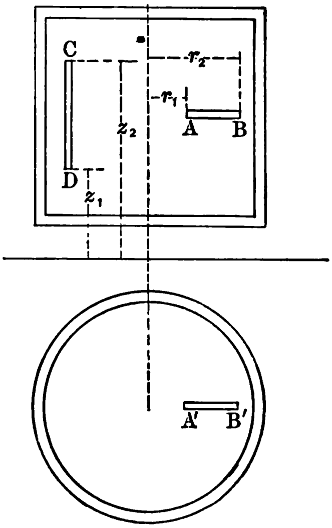


Fig. 14.

30. Variation of Pressure in Rotating Liquid. — If a body of water is forced to maintain a condition of uniform rotation about a vertical axis, the pressure increases with the distance from the axis of rotation in a manner which can be estimated as follows:

Let Fig. 14 represent horizontal and vertical sections of a cylindrical vessel filled with water. If the vessel is caused to rotate uniformly about its axis of figure, the water will soon take up a rotational motion, and the vessel and the water will rotate together as if forming one rigid body.

In order to determine the variation of pressure with the distance from the axis of rotation, consider a prismatic element of water of uniform cross-section F whose axis is horizontal and coplanar with the axis of rotation. Such an element is represented in vertical projection at AB and in horizontal projection at $A'B'$.

Let r_1 = distance of (A, A') from axis of rotation;

r_2 = " " (B, B') " " " "

p_1 = intensity of pressure at point (A, A');

p_2 = " " " " " (B, B');

ω = angular velocity of rotation.

The fundamental equation of dynamics,

$$\text{force} = \text{mass} \times \text{acceleration},$$

applies to the motion of any body whatever,* *force* meaning the resultant of all external forces acting on the body, *mass* the total mass of the body, *acceleration* the acceleration of its mass-center. Applying this to the elementary prism AB, it is seen that the acceleration of the mass-center and the resultant force have the direction BA, since the mass-center describes a circle whose center lies in the axis of rotation. The radius of this circle being $\frac{1}{2}(r_1+r_2)$, the acceleration has the value

$$\tfrac{1}{2}(r_1+r_2)\omega^2.$$

The only forces not perpendicular to AB are the pressures on the ends of the element, and the resultant of these is

$$(p_2-p_1)F.$$

The mass of the element is

$$\frac{wF(r_2-r_1)}{g}.$$

The dynamical equation therefore reduces to the form

$$\frac{p_2}{w}-\frac{p_1}{w}=\frac{(r_2{}^2-r_1{}^2)\,\omega^2}{2g}. \quad \ldots \ldots \quad (1)$$

The variation of the pressure in the vertical direction follows the same law as if the body of water were at rest. Thus, consider a prismatic element CD (Fig. 14), whose axis is vertical. If force and acceleration be resolved vertically, it is seen that, since the acceleration in the vertical direction is 0, the sum of the vertical components of all forces acting upon the element must be 0. The forces having vertical components are the normal pressures on the ends of the prism and the weight of every particle of the water. If z_1 and z_2 are the heights of D and C respectively above any horizontal plane, p_1 and p_2 the corresponding values of the pressure-intensity, and F the cross-section of the element, we have for the total downward force

$$(p_2-p_1)F+w(z_2-z_1)F.$$

* Theoretical Mechanics, Art. 380.

Equating this to 0,

$$\frac{p_2}{w}-\frac{p_1}{w}=z_1-z_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The results expressed by equations (1) and (2) may be combined into a single equation as follows:

Let A and B be any two points in the rotating liquid; z_1, z_2 their ordinates from some horizontal plane; r_1, r_2 their distances from the axis of rotation; p_1, p_2 the pressures at the two points respectively. Then

$$\frac{p_2}{w}-\frac{p_1}{w}=z_1-z_2+\frac{(r_2^2-r_1^2)\omega^2}{2g}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or, in symmetrical form,

$$z_1+\frac{p_1}{w}-\frac{r_1^2\omega^2}{2g}=z_2+\frac{p_2}{w}-\frac{r_2^2\omega^2}{2g}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It is not difficult to show that this equation holds even if the axis of rotation is not vertical. In such a case the value of z for a given particle will continually change, since it must be measured from a horizontal plane, irrespective of the direction of the axis of rotation. The form of the containing vessel is obviously of no consequence.

31. Form of Free Surface of Rotating Liquid.—It may be shown that, if a body of liquid rotates uniformly about a vertical axis, the upper surface, if free, will assume the form of a paraboloid of revolution.

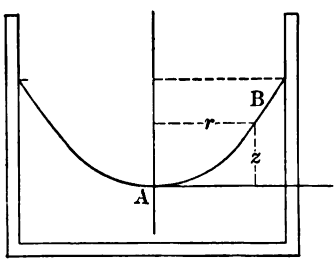


FIG. 15.

Let A (Fig. 15) be the point in which the axis of rotation pierces the free surface, and B another point of the free surface. Let z denote the height of B above a horizontal plane through A, and r the distance of B from the axis of rotation. Then in equation

(4) we may put $z_1 = 0$, $z_2 = z$, $p_1 = p_2 = 0$, $r_1 = 0$, $r_2 = r$; and the equation becomes

$$z = \frac{\omega^2 r^2}{2g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Evidently r, z are the coordinates of the curve cut from the free surface by a plane containing the axis of rotation. Equation (5) represents a parabola with vertex at A and principal diameter vertical.

Obviously the same equation results if the pressures at A and B have any equal values; all surfaces of equal pressure are therefore alike.

EXAMPLES.

1. A body of water rotates uniformly about a vertical axis, making 80 revolutions per minute. If the upper surface is free, what is the difference in level between its lowest point and a point 16 inches from the axis? *Ans.* 1.95 ft.

2. Does the variation of pressure due to rotation depend upon the density of the liquid? What is the result of Ex. 1 if the liquid is mercury (specific gravity 13.6)?

CHAPTER III.

FLOW OF WATER THROUGH ORIFICES. TORRICELLI'S THEOREM.

32. Stream of Water with Steady Flow.—The streams of water with which hydraulic discussions and experiments are concerned may be either confined in pipes, partly confined in open channels, or wholly unconfined.

If, at every cross-section of a stream, the velocity of flow and the form and size of the cross-section remain constant (the conditions at different sections, however, not necessarily being alike), the flow is said to be *steady*.

The most important practical cases are of the kind thus described, and to such the discussion will for the most part be restricted.

33. Rate of Discharge.—By *rate of discharge* of a stream is meant the quantity of water passing a given cross-section per unit time. It will usually be expressed in cubic feet per second.

It is evident that, so long as the condition of flow remains steady, the rate of discharge has the same value at all cross-sections, as well as a constant value at any given section.

34. Velocity in a Cross-section.—It is impossible to determine, either in magnitude or in direction, the velocities of all the various particles which, at any instant, are passing a given cross-section of a stream. Even at a section where the stream is neither converging nor diverging (as at A, Fig. 16), the directions of motion of the different particles doubtless differ, at least slightly, from the direction of the axis of the stream,

although the predominating motion has that direction. Still more important, probably, are the irregularities in the magnitudes of the velocities; especially in case of a confined stream, in which the particles adjacent to the confining surface are retarded by friction.

In a section where the stream converges or diverges (as at B or D) the variation in direction of the velocities of particles in different parts of the cross-section is doubtless still more important.

The component of velocity in the direction of the axis of the stream is the only component usually considered. In the following discussions, therefore, it is commonly to be understood that "velocity" means "axial component of velocity."

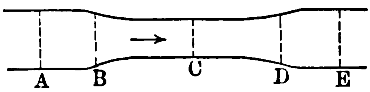


FIG. 16.

35. Mean Velocity in a Cross-section.—Mean velocity in a cross-section may be defined as the quotient of the rate of discharge by the area of the cross-section.

Let q = rate of discharge (cu. ft. per sec.);
F = area of cross-section (sq. ft.);
v = mean velocity in cross-section (ft. per sec.);

then $$v=\frac{q}{F}, \quad \text{or} \quad q=Fv.$$

36. Values of Mean Velocity in Different Cross-sections.—In case of steady flow, the mean velocity remains constant at any given cross-section, but has different values at different sections if the cross-sectional areas are unequal. If the values of the area at different sections are denoted by $F_1, F_2, \ldots,$ and the corresponding values of the mean velocity by $v_1, v_2, \ldots,$ we have

$$F_1v_1=F_2v_2= \ldots =q=\text{constant}.$$

The mean velocity is thus inversely proportional to the area of cross-section.

The above equation is called the *equation of continuity.*

37. Torricelli's Theorem.—If a small orifice be opened in the side of a vessel containing water, the velocity of the escaping jet will be nearly equal to the velocity acquired by a body falling freely from rest through a vertical distance equal to the depth of the orifice below the free surface of the water.

Let h denote the depth of the orifice below the free surface, and v the velocity of the jet; then the proposition states that the following equation is nearly satisfied:

$$v = \sqrt{2gh};$$

the actual value of v being a little less than that given by the equation.

The truth of the proposition is known from experiment, and it is inferred that, if frictional resistances could be eliminated, the equation $v^2 = 2gh$ would be exactly satisfied. It will be shown later that this conclusion follows from the principle of energy.

The depth of the center of a small orifice below the free surface of the water is called the *head on the orifice.*

38. Actual Velocity of Jet.—The actual mean velocity of the jet is always less than that computed from the formula $v^2 = 2gh$; how much less depends upon the nature of the orifice.

In Fig. 17 are represented four cases. The orifices are all supposed to be circular, the smallest diameters being equal, and all are supposed to be under the same head.

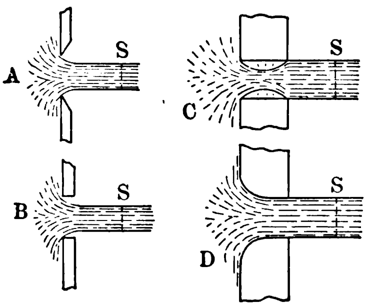


FIG. 17.

In case of the sharp-edged orifice shown at A, the stream converges as it passes the plane of the orifice. The smallest cross-section of the jet is found at S, and at this section the mean axial velocity has its greatest value. For such an orifice the value of the mean velocity is but slightly less than the value computed from the above formula. This is because

the frictional resistances (which depend upon the velocities with which adjacent portions of fluid slide over each other or over other bodies) are in this case comparatively small. Within the vessel the velocities of the particles are small, and the only place where the friction becomes important is near the orifice. The surface of contact between the jet and the vessel is so small that the retarding effect is slight.

A cylindrical orifice in a plate, such as is shown at *B*, Fig. 17, gives practically the same result as the sharp-edged orifice at *A*, if the thickness of the plate is small. If, however, the thickness is increased, the condition of the jet changes materially.

Thus, consider a cylindrical orifice in a thick plate, as shown at *C*, Fig. 17. As the jet enters the orifice it tends to converge (as in preceding cases). The air surrounding the jet at the contracted portion within the orifice is, however, quickly carried out by friction, thus reducing the pressure below that of the atmosphere; this causes the stream to expand and fill the orifice. Whether this result is complete or partial will depend upon the thickness of the plate. The frictional resistances between the particles of water (due to the irregular motions set up within the orifice), and also the friction against the cylindrical surface of the orifice, are considerably greater in this case than in the preceding, and the velocity of the jet is correspondingly less.

In case of an orifice with rounded inner edge (*D*, Fig. 17), the frictional resistance due to the surface of the orifice is greater than in the case of the sharp-edged orifice, because the area of contact is greater. The internal friction due to irregularities of motion of the particles is, however, less than in case *C*. The velocity of the jet in the case shown at *D* is greater than in case *C*, but less than in cases *A* and *B*.

In all cases, the "velocity of the jet" is taken to mean the average velocity in a section outside the orifice at a point where the jet has become cylindrical (sections such as are marked *S* in the four cases shown in Fig. 17).

39. Rate of Discharge from Small Orifice.—The rate of discharge from an orifice is equal to the product of any cross-sectional area of the stream into the mean axial velocity in that cross-section. Although this is true for any section, it is common to consider the section at which the stream has become cylindrical.

If the four cases above discussed be compared with reference to the rate of discharge (the smallest cross-section of the orifice having the same value in all cases), it will be seen that the greatest discharge does not necessarily accompany the greatest velocity. Thus, experiment shows that case D gives the greatest discharge and case C the next greatest. The reason is that the greater size of the jet at S in these cases more than counterbalances the greater velocity found in cases A and B.

40. Coefficient of Velocity.—The factor which, applied to the ideal velocity $\sqrt{2gh}$, gives the true mean velocity of the jet is called the *coefficient of velocity.* This coefficient is an abstract number less than unity.

41. Coefficient of Contraction.—The ratio of the area of the cross-section of the jet to that of the orifice is called the *coefficient of contraction.* This is also an abstract number, and in most cases of practical importance its value is less than unity.

42. Ideal Velocity and Discharge.—Practical formulas for velocity and discharge from small orifices are obtained by applying experimental coefficients to formulas which would hold in a certain ideal case. This ideal case is one in which there are supposed to be no frictional resistances to affect the velocity, and in which the cross-section of the jet is supposed to be equal to that of the orifice.

If F denotes the area of the orifice and h the head on its center, the ideal velocity is given by the formula

$$v = \sqrt{2gh},$$

and the ideal discharge (per unit time) by the formula

$$q = F\sqrt{2gh}.$$

43. Actual Velocity and Discharge.—If c' denotes the coefficient of velocity and c'' that of contraction, the true values of the mean velocity of the jet and the rate of discharge may be written

$$v = c'\sqrt{2gh};$$
$$q = c''Fv = c'c''F\sqrt{2gh}.$$

44. Coefficient of Discharge.—The ratio of the actual value of the rate of discharge to its ideal value is called the *coefficient of discharge.*

If the value of this coefficient is c, we have from the definition

$$q = cF\sqrt{2gh},$$

and therefore

$$c = c'c''.$$

45. Circular Standard Orifice.—An orifice with sharp edge (as A, Fig. 17), or an orifice in a thin plate (as B, Fig. 17), is called a *standard* orifice. The values of the coefficients of velocity and contraction for small circular orifices of this kind have been fairly well established by experiment. The following may be taken as sufficiently near the true values:

$$c' = 0.98; \quad c'' = 0.62; \quad c = c'c'' = 0.61.$$

46. Discharge from Large Orifice.—If an orifice is not small in comparison with the head on its center, it may be necessary, in estimating the discharge, to take account of the different values of the head for different parts of the cross-section. For the ideal case described in Art. 42 the value of the discharge per unit time may be found as follows:

Let dF = area of a differential element of the cross-section;
z = head on element dF;

then the discharge per unit time through this element is $\sqrt{2gz} \cdot dF$,

and the discharge per unit time for the whole orifice is the integral of this expression for the entire area F.

For the actual case, the rate of discharge is found by applying a coefficient to this ideal value. The result may be written

$$q = c\sqrt{2g}\int z^{\frac{1}{2}}\, dF.$$

47. Large Horizontal Orifice.—If the plane of the orifice is horizontal, z is constant and equal to h. The formula therefore reduces to

$$q = cF\sqrt{2gh},$$

identical in form with the formula for the case of a small orifice. The value of c is found to vary with the form of the orifice; and for circular orifices it doubtless differs somewhat from the value applying to very small orifices.

48. Large Vertical Orifice.—In order to apply the above general formula to the case in which the plane of the orifice is not horizontal, the form and dimensions of the orifice must be given. The most important case practically is that of a rectangular orifice with one pair of edges horizontal.

49. Rectangular Orifice.—Consider a rectangular orifice of horizontal width b and depth d (Fig. 18).

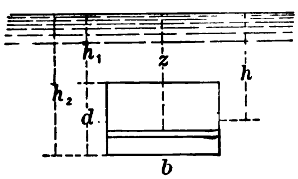


FIG. 18.

Let h_1 = head on upper edge;
h_2 = head on lower edge;
h = head on center.

To apply the general formula of Art. 46, let the differential area be an elementary strip of length b (horizontal) and width dz (vertical). Then

$$q = c\sqrt{2g}\, b\int_{h_1}^{h_2} z^{\frac{1}{2}}\, dz = c\tfrac{2}{3}\sqrt{2g}\, b(h_2^{\frac{3}{2}} - h_1^{\frac{3}{2}}).$$

If the head on the center is great in comparison with the

vertical dimension of the orifice, a simpler approximate expression may be used. Thus we have

$$h_2 = h + \frac{d}{2}; \quad h_1 = h - \frac{d}{2}.$$

Expanding $(h + d/2)^{\frac{3}{2}}$ and $(h - d/2)^{\frac{3}{2}}$ by the binomial theorem and substituting in the above formula, the result becomes

$$q = cbd\sqrt{2gh}\left[1 - \frac{1}{96}\left(\frac{d}{h}\right)^2 - \ldots\right],$$

in which the series converges rapidly except for relatively large values of d/h. If all terms except the first be neglected, we have

$$q = cbd\sqrt{2gh},$$

which is identical with the formula for discharge through a small orifice.

Whatever the form of the orifice, the formula obtained by taking account of the actual head on every part of the orifice differs little from that obtained by using the head on the centroid as applying to all parts of the area, if h exceeds a small multiple of the vertical dimension of the orifice.

50. Rectangular Notch or Weir.—In practical Hydraulics the most important case of rectangular orifice is that in which the upper side is open. The formula for this case is obtained by putting $h_1 = 0$. Thus, using a coefficient of discharge, and replacing h_2 by H, the actual value of q may be written

$$q = c\tfrac{2}{3}\sqrt{2g}\,bH^{\frac{3}{2}}.$$

51. Triangular Notch.—Another case of some importance is that of a triangular notch (Fig. 19).

Let b = width of notch at water surface;

H = head on vertex.

Take as elementary area a horizontal strip at depth z, of vertical width dz. The length of this strip is $b(H - z)/H$, and

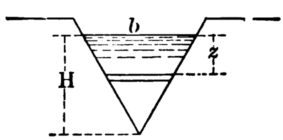


FIG. 19.

$$dF = \frac{b}{H}(H - z)dz.$$

The value of the rate of discharge for the whole area is

$$q = c\sqrt{2g}\frac{b}{H}\int_0^H (H-z)z^{\frac{1}{2}}dz,$$

or

$$q = \frac{4}{15}c\sqrt{2g}\,bH^{\frac{3}{2}},$$

c being the coefficient of discharge.

The values of the coefficient of discharge for this case and the preceding cases will be considered in Chapter XIII, in which the use of orifices and weirs for the measurement of rate of discharge is discussed.

52. Unequal Pressures on Water-surface and Jet.—In the ordinary experiments upon which Torricelli's theorem (Art. 37) is based, the surface of the water is under atmospheric pressure, and the jet discharges into the atmosphere. The theorem still holds if the pressures at these two points differ from that of the atmosphere, so long as the two are equal. If, however, unequal pressures exist at the water surface and the point of discharge, the theorem must be modified.

Consider first the case in which the pressure at the water surface exceeds that of the atmosphere by p_1, while the jet discharges into the atmosphere. Thus the space above the water (at A, Fig. 20) may contain compressed air. Evidently the effect of the pressure p_1 on the conditions existing within the body of water is the same as that of an additional depth of

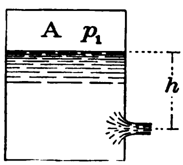


FIG. 20.

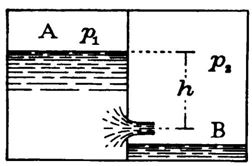


FIG. 21.

water. The height of water column necessary to produce a pressure of intensity p_1 is p_1/w; hence the discharge takes place as if under a head $h+p_1/w$ with equal pressures at the water surface and jet.

Next suppose the discharge to take place into a closed chamber (Fig. 21) in which the pressure exceeds that of the atmosphere by p_2. The effect of this pressure p_2 is the same as that of a diminution of the head on the orifice by the amount p_2/w.

It appears, therefore, that Torricelli's theorem may be applied to the case of unequal pressures if the head on the orifice be corrected for the difference of the pressures. Thus, if the pressures at A and B (Fig. 21) are p_1 and p_2 respectively, the value of the ideal velocity is

$$v=\sqrt{2g\left(h+\frac{p_1}{w}-\frac{p_2}{w}\right)}.$$

EXAMPLES.

1. If the depth of the orifice below the water surface (Fig. 21) is 4 ft., the pressure at A atmospheric, and that at B absolute zero, compute the velocity of the jet. What head would produce the same velocity if the pressures at A and B were equal? *Ans.* $v=49.4$ ft. per sec.

2. If in Fig. 21 $h=40$ ft., $p_1=$absolute zero, $p_2=$atmospheric pressure, compute the velocity of the jet.

53. Submerged Orifice.—If the discharge takes place as in Fig. 22, the orifice being below the surface of the water in the receiving chamber, the jet at B is under a pressure wh_2 in addition to the pressure existing at the surface D. The head on the orifice must therefore be corrected by the subtraction of h_2, in applying Torricelli's theorem. That is, the ideal velocity is

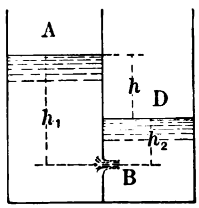


FIG. 22.

$$v=\sqrt{2g(h_1-h_2)}.$$

In applying this formula to actual cases, the coefficients of velocity and discharge must be given different values from those applying to similar orifices in case of discharge into air.

The actual values of coefficients of discharge for some of

the cases of most practical importance will be given in Chapter XIII.

The foregoing principles will receive theoretical justification in the following chapters, in which the theory of energy is applied to all cases of steady flow.

54. Discharge under Varying Head.—If the head on an orifice varies, the total discharge in a given time, or the time required for a given total discharge, can be computed only by integration.

Let the head on the orifice at any instant be y, the rate of discharge at that instant being q, and let c be the coefficient of discharge; then

$$q = cF\sqrt{2gy}.$$

If dQ denotes the volume discharged in the time dt,

$$dQ = q\,dt = cF\sqrt{2gy}\cdot dt.$$

In a particular case dQ can be expressed in terms of y and dy and the equation can be integrated.

55. Time of Emptying a Reservoir.—Consider a vessel or reservoir of any form, filled with water to a certain level, and let it be required to determine the total time required for the surface to fall any given amount. It will be assumed that the coefficient of discharge does not vary with the head. (See Fig. 23.)

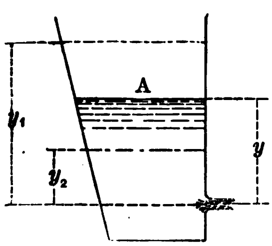


Fig. 23.

Let y = head on center of orifice at time t;

y_1 = initial value of y;
y_2 = final value of y;
A = horizontal cross-section of reservoir at water surface (variable);

F = area of orifice;
Q = total volume discharged from some assumed instant up to the instant t;
q = rate of discharge at time t;
c = coefficient of discharge, assumed constant.

Then, as in the preceding article,

$$dQ = q\,dt = cF\sqrt{2gy}\cdot dt.$$

But also

$$dQ = -A\,dy$$

(the minus sign being used because Q increases as y decreases); hence

$$cF\sqrt{2gy}\cdot dt = -A\,dy.$$

If A is variable, it must be known as a function of y in order that the solution may be completed.

Reservoir of uniform horizontal cross-section.—Let A be constant, and let T denote the time required for y to change from y_1 to y_2. Integrating the last equation between the stated limits,

$$\frac{cF\sqrt{2g}}{2A}T = y_1^{\frac{1}{2}} - y_2^{\frac{1}{2}}.$$

56. Time of Filling a Reservoir.—If water flows from one reservoir or chamber into another through a submerged orifice, the pressure against which the flow takes place increases as the water surface rises in the receiving chamber. If the horizontal area of the supply reservoir is great in comparison with that of the receiving chamber, the drop in the surface of the former will be inappreciable. In this case let y denote the difference in level between the water surfaces in the two reservoirs at any instant, the remaining notation being as

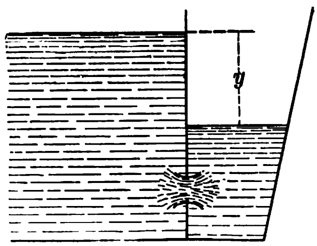


FIG. 24.

in the preceding case. Then the reasoning of Art. 55 applies without change, leading to the same formula.

57. Canal Lock.—A canal lock consists of a chamber or compartment which is placed in communication alternately with two bodies of water whose surfaces are at different levels. The time of emptying or filling the lock may be estimated by the above formula, with proper value of c.

If the discharge into or from the lock takes place through a tunnel, the coefficient of discharge will be much less than for a simple orifice. Its value will depend upon the length of the tunnel, the form and size of its cross-section, and the roughness of its surface. The problem will be analogous to that of estimating the effect of friction on the flow in a pipe, to be discussed in Chapter IX.

EXAMPLES.

1. A reservoir of 1000 sq. ft. horizontal cross-section is emptied through an orifice of area 2 sq. ft. Taking the head on the center of the orifice as initially 19 ft., compute the times required for the surface to drop 1 ft., 3.5 ft., and 7 ft. respectively. Take $c=0.65$.
Ans. 22.2 sec.; 81.0 sec.; 172 sec.

2. With initial conditions as in Ex. 1, compute the drop of the water surface in 3.5 min. and in 7 min. *Ans.* 8.3 ft.; 14.3 ft.

3. Two reservoirs of horizontal cross-sections A', A'', are connected by a submerged orifice. If the difference in level of the two water surfaces at any instant is denoted by y, show that the time required for y to change from y_1 to y_2 is given by the formula

$$\frac{cF\sqrt{2g}}{2A}T = y_1^{\frac{1}{2}} - y_2^{\frac{1}{2}},$$

in which $A = A'A''/(A'+A'')$. This includes the two cases above treated, reducing to one of them if A' or A'' is infinite.

4. As a particular case of Ex. 3, let $A'=24$ sq. ft., $A''=10$ sq. ft., $F=0.5$ sq. ft., $y_1=15$ ft., $y_2=0$, $c=0.6$. Determine T.

CHAPTER IV.

THEORY OF ENERGY APPLIED TO STEADY STREAM MOTION.*

58. Transformation and Transference of Energy in Steadily Flowing Stream.—Let AB (Fig. 25) represent a portion of a steady stream, the direction of flow being from A toward B. Consider the transformations and transferences of energy in which any given particle of water is concerned.

The energy possessed by a particle at any instant is in general part potential and part kinetic.

A particle of mass m having velocity v possesses $mv^2/2$ units of kinetic energy. If v is in feet per second and m in engineers' kinetic units † (the unit mass being equal to 32.2 pounds-mass), $mv^2/2$ is in foot-pounds.

In estimating potential energy due to gravity a horizontal reference plane must be chosen. If a particle weighs W pounds and is z feet above the reference plane, it possesses Wz foot-pounds of potential energy.‡

Thus, the potential energy of any particle of water moving with the stream increases or decreases with the height of the particle above datum, and its kinetic energy increases or decreases with its velocity. It is instructive to consider somewhat definitely how these changes occur in the case of steady flow.

(1) Any small portion of the fluid is continually receiving energy from certain adjacent particles and giving up energy to

* The theory of steady flow of gases is treated in Appendix A.

† Theoretical Mechanics, Art. 218.

‡ Strictly it should be said that this potential energy is possessed by the system consisting of the body and the earth. See Theoretical Mechanics, Art. 362.

others. Thus, consider a small body of the water between two transverse planes (as X, Fig. 25). This body X is acted upon by pressures exerted by the adjacent bodies of water P and Q. Let F' be the area of the cross-section between P and X, and p' the normal pressure per unit area exerted across this section; let F'' be the area of the cross-section between X and Q, and p'' the intensity of pressure in this section; and let v', v'' be the velocities in the cross-sections F', F'' respectively.* In a short interval of time Δt the body X receives an amount of energy $p'F'v'\Delta t$ by reason of the positive work done upon it by the total pressure $F'p'$; the body P loses an equal quantity of energy because an equal amount of negative work is done upon it. During the same time the body X loses an amount of energy $p''F''v''\Delta t$ because of the negative work done upon it by the pressure $p''F''$; the body Q gaining an equal quantity of energy. In other words, during the time Δt the body X receives from P a quantity of energy $p'F'v'\Delta t$, and gives up to Q a quantity $p''F''v''\Delta t$. Since $F'v' = F''v''$, these two amounts of energy will be equal if $p' = p''$; hence in that case the body X neither gains nor loses energy by reason of the pressures, but acts as a transmitter of energy † from P to Q. If, however, p' and p'' are unequal, the body X receives from P either more or less energy than it gives to Q. But in any case, so far as the process here considered is concerned, one portion of water gains exactly as much energy as other parts lose; the energy possessed by the whole stream remains constant, while that of individual particles continually varies.

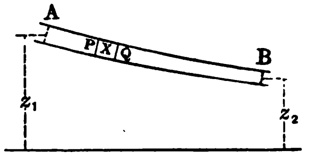


FIG. 25.

(2) If two adjacent portions of the water slide over one another, the tangential forces which resist such sliding result in a twofold energy change. So far as the two portions have

* The velocity is assumed uniform throughout each cross-section. If this is not true, the reasoning still holds for an elementary portion of the stream.

† This is analogous to the transmission of energy from one pulley to another through a belt.

a common component of motion parallel to the sliding, there is a transfer of energy from one portion to the other exactly similar to that just explained as due to normal pressure. But since the two portions move unequally in the stated direction, the energy lost by one is not equal to that gained by the other. The negative work done on one portion is always greater than the positive work done on the other, so that the net result is a loss of energy. The mechanical energy thus lost is transformed into molecular energy or heat. This transformation of mechanical energy into heat is called dissipation of energy. Energy thus dissipated is practically lost, so far as the possibility of its utilization is concerned.

(3) Particles adjacent to the surface of the pipe (or whatever body encloses the stream) lose energy by reason of the negative work done upon them by the frictional forces exerted by the pipe surface. The energy thus lost is dissipated into heat.

It is thus seen that in any steady stream there is in general a continual *transference* of energy in the direction of the flow, accompanied by a *dissipation* of mechanical energy into heat. Because of this dissipation, the energy transferred across a section A (Fig. 25) is always greater than that transferred across a section B down-stream from A. This will be made definite in the following article.

59. Energy Passing Any Given Section.—A useful expression may be deduced for the total quantity of energy passing any cross-section of a steady stream in a given time.

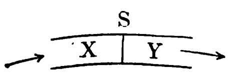


FIG. 26.

Consider any section of the stream, as S (Fig. 26).

Let p = intensity of pressure at every point in the section (lbs. per sq. ft.);

F = area of cross-section (sq. ft.);

z = ordinate of centroid of section (ft.) above a horizontal datum plane, taken as reference plane in estimating potential energy;

v = velocity of flow across the section (ft. per sec.);

W = weight of water passing the section per unit time (lbs. per sec.);

w = weight of unit volume of water (lbs. per cu. ft.);

$\frac{W}{w}$ = volume of water passing the section per unit time (cu. ft. per sec.).

Energy passes the section S in two ways: (*a*) Each particle of water passing the section possesses a certain quantity of energy, part potential and part kinetic; (*b*) the particles on one side of the section are at every instant giving up energy to the particles on the other side by reason of the pressure and motion.

(*a*) During a short interval of time Δt, $W\Delta t$ pounds of water pass the section. This water possesses potential energy of amount $W\Delta t \cdot z$, and kinetic energy of amount $\frac{W\Delta t \cdot v^2}{2g}$; i.e., a total quantity of energy

$$(W\Delta t)\left(z+\frac{v^2}{2g}\right).$$

(*b*) Designating by X and Y the bodies of water adjacent to the section and separated by it (Fig. 26), it is seen that, by reason of the pressure exerted by X and Y upon each other, the body X loses and the body Y gains a quantity of energy which may be computed as follows: The total pressure resisting the motion of X is Fp; an equal and opposite force acts upon Y in the direction of the motion. In a time Δt, the bodies upon which these forces act move a distance $v\Delta t$. The force Fp acting upon X does an amount of work $-Fpv\Delta t$, and the force Fp acting upon Y does an amount of work $+Fpv\Delta t$. That is, X loses a certain amount of energy and Y gains an equal amount; or there is a transfer from X to Y of an amount of energy

$$Fpv\Delta t.$$

Since $W = wFv$, the quantity of energy passing the section in time Δt by reason of the transfer of energy from X to Y may

be written in either of the equivalent forms

$$Fpv\Delta t=\frac{W}{w}p\Delta t=W\Delta t\frac{p}{w}.$$

Combining the values found in paragraphs (*a*) and (*b*), there results for the total energy passing the section S in the time Δt the value

$$W\Delta t\left(z+\frac{v^2}{2g}+\frac{p}{w}\right).$$

The quantity of energy passing the section *per unit time* is evidently

$$W\left(z+\frac{v^2}{2g}+\frac{p}{w}\right).$$

Stated in still another way, it is seen that *for every unit weight of water* passing any section a quantity of energy

$$z+\frac{v^2}{2g}+\frac{p}{w}$$

passes the same section. The quantities of water passing different sections in any time are equal, but the quantities of energy are unequal, because p and v in general change from section to section.

60. General Equation of Energy in Steady Flow.—An important equation, often called Bernoulli's theorem, is obtained by applying the theory of energy to a portion of a steadily flowing stream.

Referring to Fig. 25, consider the volume included between two fixed cross-sections A and B. Let pressure, velocity, area of cross-section, height above datum, etc., be represented by the same symbols as above, with suffix ($_1$) for the up-stream section A and ($_2$) for the down-stream section B.

If the flow is steady, it is evident that the total quantity of mechanical energy contained in the volume AB remains

constant. For, considering any elementary volume in a fixed position in the stream, the water occupying it at any instant has the same mass and velocity as that which has replaced it at any succeeding instant, and the two elements therefore possess equal quantities of energy, both kinetic and potential. The total mechanical energy gained by the volume *AB* during any time must therefore equal the total mechanical energy lost by it during the same time.

Consider the energy gained and lost by the volume *AB* while W pounds of water pass any section of the stream.

The only gain of mechanical energy is that passing the section A; its value is

$$W\left(z_1+\frac{v_1^2}{2g}+\frac{p_1}{w}\right).$$

The mechanical energy lost is made up of two parts,—that passing the section B, and that dissipated into heat. The value of the former part is

$$W\left(z_2+\frac{v_2^2}{2g}+\frac{p_2}{w}\right).$$

The value of the energy dissipated will be represented by K.

Equating the total energy gained by the volume *AB* to the total energy lost by it, we have

$$W\left(z_1+\frac{v_1^2}{2g}+\frac{p_1}{w}\right)=W\left(z_2+\frac{v_2^2}{2g}+\frac{p_2}{w}\right)+K,$$

or

$$z_1+\frac{v_1^2}{2g}+\frac{p_1}{w}=z_2+\frac{v_2^2}{2g}+\frac{p_2}{w}+H', \quad . \quad . \quad . \quad . \quad (1)$$

in which $H'=\frac{K}{W}$.

Equation (1) expresses Bernoulli's theorem, or the general equation of energy for steady flow.

61. Meaning of " Head."—The word head is used by writers on Hydraulics in a somewhat indefinite way. In all cases it

means the height of a column of water, either actual or ideal. Thus, "head on an orifice," or on any point of an orifice, has in preceding discussions been used to designate the vertical height of the free surface of water above the point under consideration.

Most writers at the present time use the word head in the following senses:

The *pressure head* at any point in a body of water is the height of a column of water which, in equilibrium, would by its weight produce the pressure existing at the point. The pressure head corresponding to a pressure of intensity p is therefore equal to p/w.

The *velocity head* at any point of a stream is the height through which a body must fall from rest under gravity to acquire a velocity equal to that existing at the given point. The velocity head corresponding to a velocity v is therefore $v^2/2g$.

The sum of the pressure head and the velocity head at any point of a stream is often called the *effective head* at that point. It seems desirable, however, to modify the meaning of this term in the following manner.

62. Effective Head.—It has been shown (Art. 59) that the total energy passing a given section of a steady stream, per unit weight of water discharged, is equal to

$$z+\frac{v^2}{2g}+\frac{p}{w}.$$

Each of the three terms in this expression represents a linear magnitude, and each may be called a "head." The second and third terms may be called *velocity head* and *pressure head* respectively, in accordance with usage as explained above. The term z has been called *elevation head*, and also *potential head*. The former term will be here adopted.

Effective head will here be defined as the sum of the pressure head, velocity head, and elevation head.

It seems desirable to include the term z in the definition of effective head, for the reason that this term has equal signifi-

cance with the other two in estimating the total energy delivered at a given point in the stream.

The symbol H will hereafter be employed to designate the value of the effective head at any point of a stream. That is,

$$H=z+\frac{v^2}{2g}+\frac{p}{w}.$$

The values of H at different cross-sections, like those of z, p, and v, will be distinguished by suffixes.

63. Lost Head.—The general equation of energy (Art. 60) shows that the effective head decreases along the stream in the direction of the flow. Thus, with the above notation, the equation may be written

$$H_1-H_2=H'.$$

The quantity H' is a linear magnitude, and expresses the amount by which the effective head decreases from A to B (Fig. 25); H' may therefore be called the *loss of head* between A and B. That is,

The loss of head between any two sections of a stream is defined as the amount by which the effective head at the up-stream section exceeds that at the down-stream section.

The reasoning by which the general equation of energy was deduced shows that H' has an important meaning as energy. Since $H'=K/W$, in which K denotes the energy lost by dissipation per unit time between A and B, and W the weight of water discharged (across any section) per unit time, it is seen that

The loss of effective head between any two sections of a steady stream is equal to the energy lost by dissipation between those sections per unit weight of water discharged.

In practical applications, the value of the lost head between two sections is sometimes found by determining H_1 and H_2 and taking their difference; and sometimes its value is estimated from a consideration of its energy meaning as just explained.

It is instructive to devote some time to the application of the equation of energy on the assumption of no loss of head. In some cases the results have practical value, the actual losses being small. In other cases such a treatment is useful only in illustrating general principles.

EXAMPLES.

2609

1. In a horizontal pipe 6 inches in diameter water flows with a velocity of 12 ft. per sec. At a section A the mean pressure is 50 lbs. per sq. in., and at a section B it is 40 lbs. per sq in. Compute the quantity of energy passing each of these sections in one second, estimating potential energy with reference to a datum plane 18 ft. below the axis of the pipe.
Ans. 19,950 foot-pounds and 16,540 foot-pounds.

2. In Ex. 1 what is the value of the lost head between A and B? In which direction is the flow?

3. The diameter of a pipe is 12 inches at a section A, 16 ft. above datum, and 8 inches at a section B, 8 ft. above datum. At A the velocity is 12 ft. per sec. and the pressure head 12.6 ft. Assuming no dissipation of energy between A and B, compute the velocity and the pressure head at B. *Ans.* Velocity = 27 ft. per sec.; pressure head = 11.5 ft.

CHAPTER V.

APPLICATION OF GENERAL EQUATION OF ENERGY, NEGLECTING LOSSES BY DISSIPATION.

64. General Method.—Assuming no loss of energy by dissipation in any part of the stream, the general equation of energy may be written

$$z+\frac{v^2}{2g}+\frac{p}{w}=H=\text{constant}.$$

The general method of applying this equation is as follows:

(1) Choose a datum plane. This fixes the value of z at every point of the stream.

(2) Notice at what points the velocity is known. If at any point the cross-section is very great in comparison with its value at other points, the velocity will be so small that the velocity head at that point may be neglected.

(3) Notice at what points of the stream, if any, the pressure is known.

(4) Notice whether there is any point of the stream at which the three parts of the effective head—elevation head, pressure head, and velocity head—are all known. If there is, the value of H becomes known at that point, and therefore at all sections of the stream.

(5) If there is no point at which elevation, pressure, and velocity are all known, note the points at which the value of H can be expressed with the use of the fewest unknown quantities. By writing expressions for H for two or more sections, it may be possible to solve the resulting equations and determine the unknown quantities.

65. Flow From a Reservoir Through an Orifice.—Consider a small orifice in the side of a vessel (Fig. 27), from which a jet is discharged into the atmosphere. Let the center of the smallest cross-section of the jet (S) be at a distance h below the level of the free surface of the water in the vessel.

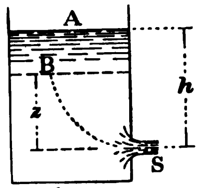


FIG. 27.

The particles of water approaching the orifice doubtless come from various parts of the reservoir; but it will appear presently that the same solution of the problem will result whatever point in the reservoir be taken as a point of the stream.

(1) Choose as datum the horizontal plane through the center of the stream at S. (Any other horizontal plane might be chosen.)

(2) The velocities of particles within the reservoir are very small, except in the neighborhood of the orifice. For any point at some distance from the orifice the velocity may therefore be taken as zero without sensible error.

(3) The jet being surrounded by the atmosphere, the pressure within the stream at S is atmospheric. Atmospheric pressure also exists at the water surface A. At points within the vessel (except near the orifice where the particles have sensible velocities) the pressure follows the hydrostatic law; that is, at a depth y below the free surface the pressure exceeds atmospheric by wy.

(4) At any point within the reservoir at which the velocity is sensibly zero the value of the effective head is known immediately. Consider a point B (Fig. 27). Whatever the value of z for this point, we have for the value of the pressure head (calling atmospheric pressure p_0)

$$\frac{p}{w}=\frac{p_0}{w}+(h-z).$$

The velocity head being sensibly zero,

$$z+\frac{p}{w}+\frac{v^2}{2g}=z+\frac{p_0}{w}+(h-z)=\frac{p_0}{w}+h.$$

That is,
$$H = \frac{p_0}{w} + h,$$

so that the value of the effective head at all points becomes known.

To complete the solution, let v denote the velocity of the jet at S. The values of elevation head, pressure head, and velocity head for this section are 0, $\frac{p_0}{w}$, and $\frac{v^2}{2g}$; that is,

$$H = \frac{p_0}{w} + \frac{v^2}{2g}.$$

Equating the values of H,

$$\frac{v^2}{2g} = h.$$

It is thus seen that, neglecting frictional losses of energy, Torricelli's theorem (Art. 37) is a direct consequence of the theory of energy.

66. Case of Unequal Pressures.—If the pressures at the surface of the reservoir and at the section S of the jet are unequal, let their values (per unit area) be p_1 and p_2 respectively. Then for the point B (Fig. 28) we have

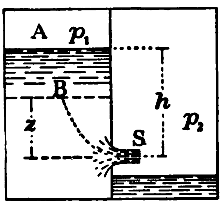


FIG. 28.

$$\frac{p}{w} = \frac{p_1}{w} + (h - z); \quad H = \frac{p_1}{w} + h;$$

and for the point S

$$H = \frac{p_2}{w} + \frac{v^2}{2g}.$$

Equating these values,

$$\frac{v^2}{2g} = h + \frac{p_1}{w} - \frac{p_2}{w},$$

which agrees with Art. 52.

67. Velocity of Approach.—If the cross-section of the reservoir is not so great that the downward velocity of the water may be neglected, another term enters the formula.

Let v' = velocity of water at point B (Fig. 27);
v = velocity of jet at section S.

Then, assuming atmospheric pressure at A and at S, we have for the point B

$$H = \frac{p_0}{w} + h + \frac{v'^2}{2g};$$

and for the point S

$$H = \frac{p_0}{w} + \frac{v^2}{2g}.$$

Equating these values of H,

$$h + \frac{v'^2}{2g} = \frac{v^2}{2g}.$$

Let F = cross-section of jet at S;
F' = cross-section of vessel at B.

Then $$F'v' = Fv, \quad \text{or} \quad v' = \left(\frac{F}{F'}\right)v.$$

Substituting this value of v' in the foregoing equation and solving for v, we have

$$v = \sqrt{\frac{2gh}{1 - \left(\frac{F}{F'}\right)^2}}.$$

If the pressures at A and S are unequal, the above reasoning leads to an equation similar to the above with $h + p_1/w - p_2/w$ substituted for h.

EXAMPLES.

1. If the diameter of the jet is 0.5″ and that of the vessel 2″, both being circular in cross-section, what percentage of error is introduced by neglecting velocity of approach? *Ans.* About 0.2 per cent.

2. For what value of the ratio F/F' will an error of one per cent be introduced by neglecting velocity of approach?

68. Effect of Velocity of Approach in Case of Rectangular Weir.—The foregoing discussion has referred to an orifice so small that no important error results from regarding the head on the center as applying to all parts of the orifice. The effect of velocity of approach in case of large vertical orifices needs special consideration. It will be sufficient to consider a rectangular orifice, since it is this form that possesses most practical importance.

Consider a stream flowing horizontally in an open channel (Fig. 29) and discharging at the end of the channel through a

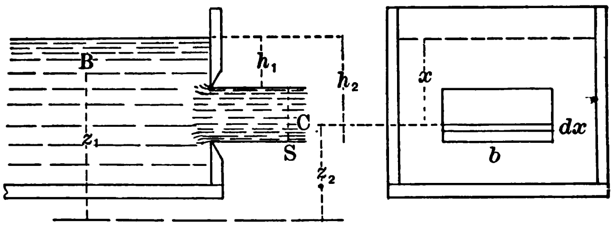


FIG. 29.

rectangular orifice of width b, the upper and lower edges being at depths h_1 and h_2 respectively below the surface of the water in the channel. Consider the ideal case in which there is no contraction of the stream and no loss of energy by dissipation, and assume that the velocity of the approaching stream has the same value throughout all parts of any given cross-section. Let v' be the value of this velocity.

At the cross-section S of the issuing stream, let v = velocity at a point C whose depth below the water surface in the channel is x. The value of v is found by applying the equation of energy to the point C and a point B of the approaching stream.

Let z_1 and z_2 denote the ordinates of B and C respectively above any assumed datum; then the depth of B below the water surface is $x+z_2-z_1$, and the pressure head at B is

$$x+z_2-z_1+\frac{p_0}{w},$$

p_0 being atmospheric pressure. Hence

$$H=z_1+\left(x+z_2-z_1+\frac{p_0}{w}\right)+\frac{v'^2}{2g}=x+z_2+\frac{p_0}{w}+\frac{v'^2}{2g}.$$

For the point C, the pressure being that of the atmosphere, we have

$$H = z_2 + \frac{p_0}{w} + \frac{v^2}{2g}.$$

Equating these values of H,

$$\frac{v^2}{2g} = \frac{v'^2}{2g} + x.$$

Taking an elementary area of length b and width dx, we have for the discharge per unit time through that element

$$dq = bv\,dx = b(v'^2 + 2gx)^{\frac{1}{2}} dx.$$

Integrating,

$$q = b\sqrt{2g}\int_{h_1}^{h_2}\left(\frac{v'^2}{2g} + x\right)^{\frac{1}{2}} dx$$

$$= \tfrac{2}{3}\sqrt{2g}\cdot b\left[\left(\frac{v'^2}{2g} + h_2\right)^{\frac{3}{2}} - \left(\frac{v'^2}{2g} + h_1\right)^{\frac{3}{2}}\right].$$

Inspection of this result shows that the velocity of approach, so far as its effect on the rate of discharge is concerned, is equivalent to an increase of $v'^2/2g$ in the head on every part of the orifice.

In case of a weir, the value of h_1 is zero. If H denotes the "head on the crest," i.e., the depth of the bottom edge (or crest) of the orifice below the surface of the approaching stream, and h' is written for $v'^2/2g$ (the head "equivalent to" the velocity of approach v'), the formula becomes

$$q = \tfrac{2}{3}\sqrt{2g}\cdot b[(H + h')^{\frac{3}{2}} - h'^{\frac{3}{2}}].$$

69. Discharge from Reservoir Through Tube or Pipe.—If a pipe or tube, or any series of pipes, leads from a reservoir and discharges a stream at any point, it is evident that the velocity of discharge (on the assumption that no energy is dissipated) is given by the same formula as that applying to discharge from an orifice.

Thus, let any series of pipes lead from a large reservoir (Fig. 30) and discharge into the atmosphere at a point whose depth below the water surface is h. The reasoning employed in the discussion of discharge through an orifice applies unchanged, and the velocity of the stream at S is given by the formula

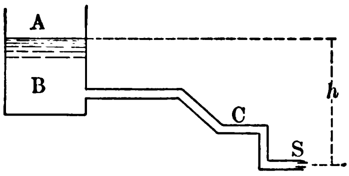


FIG. 30.

$$v = \sqrt{2gh}.$$

If the cross-section of the issuing stream is known, the rate of discharge is known from the relation

$$q = Fv.$$

70. Pressure at Any Section.—After the value of q has been determined the velocity and pressure at any point in the stream may be computed, if the cross-section at that point is known.

Thus, in Fig. 30, let $h = 40$ ft., and let the diameter of the jet at S be 1 inch. Let it be required to determine the velocity and pressure at a point C, 10 ft. higher than S, the pipe at C being 1.25 inches in diameter.

Let v_2 = velocity at S; v_1 = velocity at C; p_1 = pressure at C. As above, we have

$$\frac{v_2^2}{2g} = h = 40.$$

Taking datum plane through the center of the jet at S, we have

$$H = \frac{p_0}{w} + \frac{v_2^2}{2g} = \frac{p_0}{w} + 40.$$

For the point C,

$$H = 10 + \frac{p_1}{w} + \frac{v_1^2}{2g}.$$

Equating values of H,

$$10 + \frac{p_1}{w} + \frac{v_1^2}{2g} = \frac{p_0}{w} + 40,$$

or
$$\frac{p_1}{w}=\frac{p_0}{w}+30-\frac{v_1^2}{2g}.$$

But $\left(\frac{5}{4}\right)^2 v_1=v_2$, or $\frac{v_1^2}{2g}=\left(\frac{4}{5}\right)^4\frac{v_2^2}{2g}=16.4$; hence

$$\frac{p_1}{w}=\frac{p_0}{w}+13.6.$$

That is, the pressure at C exceeds that of the atmosphere by the equivalent of 13.6 ft. head.

It will be noticed that, if different points of the stream at the same level be compared, the pressure is greatest where the velocity is least. Since a contraction of the stream increases the velocity, it causes a decrease of pressure.

More generally, consider what is implied by the equation

$$z+\frac{p}{w}+\frac{v^2}{2g}=H=\text{constant}.$$

(1) For a stream of uniform cross-section and consequently having equal velocities at all sections, the pressure head increases as the elevation decreases, and *vice versa;* the variation of pressure follows the hydrostatic law.

(2) For a stream all parts of which are at the same level the pressure head increases as the velocity head decreases, and *vice versa.*

When losses of energy by dissipation are considered, these principles are greatly modified. But it is instructive to consider carefully the conditions in the ideal case of no loss of energy.

71. Pressure Can Never be Negative.—In such a case as that shown in Fig. 30, the velocity of the jet at S is independent of the dimensions or elevation of any part of the pipe leading from the reservoir to the point of discharge. The only conditions affecting the velocity of outflow are the pressure and elevation at A and S, and the relation between the cross-sectional areas at these points. Between these sections the pipe may

vary in diameter in any way, and its elevation may vary in any way without affecting the velocity of discharge. There is, however, an important limitation on the application of this theory, even without the consideration of losses of energy.

From the equation

$$z+\frac{p}{w}+\frac{v^2}{2g}=H=\text{constant}$$

it is evident that, by increasing v or z, the pressure p may be decreased to any assigned value. The velocity of outflow, and therefore the rate of discharge, being fixed, if at any point the cross-section is made very small the velocity becomes very great. Since $v=q/F$, there is no limit to the value which may be given to v by decreasing F. For any value of z, therefore, p/w may be made zero, or even negative, by decreasing the cross-section of the pipe.

A negative pressure would mean a tension, and the nature of a liquid is such that a tendency to tension causes separation of the particles.* Therefore if the pressure at any point of the stream, as computed by the foregoing theory, is found to be negative, the conclusion to be drawn is that the stream will break at that point, and the solution must be modified.

If, at the point where the pressure as computed from the formula has the least (algebraic) value, this value is negative, the actual pressure may be put equal to zero; and the velocity computed on this assumption may be regarded as the greatest possible value of the velocity at that point on the supposition of no dissipation of energy.

EXAMPLES.

1. In Fig. 31, let the diameter of the pipe have the following values at different points: at B, 3″; at C, 2.5″; at D, 2.5″; at E, 2″. Let the diameter of the jet at F be 1.75″. Determine the velocity and pressure at each of the four points B, C, D, E.

2. Keeping the size of the pipe unchanged, how high can the pipe be carried at D without reducing the pressure to absolute zero?

* Strictly, it should be said that under certain conditions a slight tension can exist in liquids; but the statement above made is practically true.

3. With the pipe D at the elevation shown in Fig. 31, how small may the diameter be without reducing the pressure to absolute zero? What will nappen if the pipe is made smaller than this limiting size?

Ans. Diameter at $D=1.80''$.

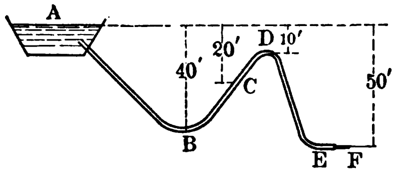


FIG. 31.

4. What will be the velocity of discharge from the siphon shown in Fig. 32? *Ans.* 11.3 ft. per sec.

5. Discuss the pressure throughout the tube in Fig. 32, assuming the diameter to be uniform. How small may the pipe be at A (as compared with its size at B) without making the pressure at A absolutely zero?

Ans. $\dfrac{\text{Diameter at } A}{\text{Diameter at } B}=0.51$ (limiting value).

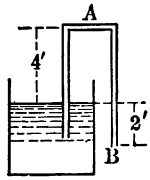


FIG. 32.

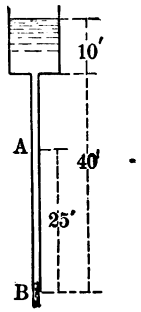


FIG. 33.

6. In Fig. 33, let the diameter of the pipe be 2″ and that of the jet at B 1.75″. Compute the pressure at the point A. At what points in the pipe has the pressure the least and greatest values?

Ans. At A, $p/w=p_0/w-4.3$.

7. In Fig. 34, let the diameter of the diverging tube be 1″ at B and 1.5″ at C, the jet at S having the same size as the tube at C. Compute velocity and pressure at B and at C.

Ans. At B, $v=51.2$; $p/w=p_0/w-32.5$.

8. In Fig. 34, if the tube were cut off at B, how would the discharge be affected?

9. In Fig. 34, what is the effect of increasing the diameter of the tube at C, that at B remaining unchanged? What is the greatest possible value of the velocity at B? *Ans.* 51.9 ft. per sec.

10. In Fig. 35, let the diameter of the pipe be 2″ at A and 1″ at B, its axis being horizontal. The pressures at the two sections are measured by the heights of water standing in the vertical tubes AA' and BB'. If the water surface is 2′ higher at A' than at B', what is the rate of discharge? *Ans.* $q = .064$ cu. ft. per sec.

11. Fig. 36 represents a pipe in which water flows in the direction AB. In order to measure the difference between the pressures at the

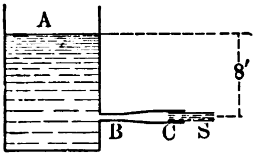


FIG. 34.

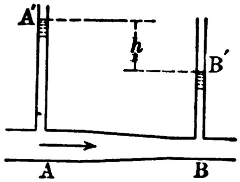


FIG. 35.

sections A and B, tubes AA' and BB' are inserted at the two sections and connected with the two branches of a U tube M, containing mercury. If mercury fills the portion of the tube $A'B'$ and water the portions AA' and BB', the difference between the heights of A' and B' indicates the difference between the pressures in the sections A and B. The specific gravity of mercury may be taken as 13.6.

(*a*) If the vertical height of B' above A' is h, what is the difference between the pressures in the tube at A' and B', expressed in "head" or height of water column? *Ans.* $13.6h$.

(*b*) What is the difference between the pressure at B' and that at A'', on a level with B'? *Ans.* $12.6h$.

(*c*) What is the difference between the values of the pressure head at C and D, these points being at the same level?

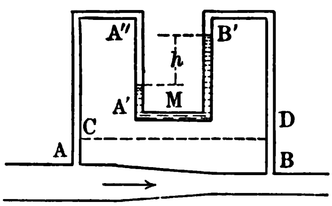


FIG. 36.

12. With the arrangement shown in Fig. 36, let the diameter at A be 3″ and that at B 2″. If $h = 6''$, what is the velocity of flow at B? *Ans.* 22.5 ft. per sec.

13. In Fig. 36, let the cross-sectional areas at A and B be F_1 and F_2, and the corresponding velocities v_1 and v_2, and let h be the difference of level of the mercury surfaces at A' and B'. Deduce the following formulas (s denotes specific gravity of mercury):

$$v_1 = \sqrt{\frac{2(s-1)gh}{\left(\frac{F_1}{F_2}\right)^2 - 1}}; \quad v_2 = \sqrt{\frac{2(s-1)gh}{1 - \left(\frac{F_2}{F_1}\right)^2}}.$$

14. In Fig. 35, let the diameters at A and B be 4″ and 3″ respectively and suppose pistons to be fitted into the pipe at the two sections, the intervening space being filled with water. If the piston at A advances at the rate of 6 ft. per sec., and the height of A' above the center of the pipe is 7′, compute the difference between the total pressures upon the two pistons. Neglect friction.

Ans. 20.9 lbs., neglecting atmospheric pressure.

CHAPTER VI.

APPLICATION OF GENERAL EQUATION OF ENERGY, TAKING ACCOUNT OF LOSSES.

72. Methods of Estimating Loss of Head.—When losses of energy by dissipation are taken into account, it is necessary to estimate the value of H' in the general equation of energy,

$$H_1 - H_2 = H'.$$

As explained in Art. 63, this term is called the loss of head between the two sections at which the effective head has the values H_1, H_2, respectively, the former referring to the up-stream section. The energy meaning of H', as stated in Art. 63, should be kept clearly in mind. It is the *energy lost by dissipation between the two sections considered, per pound of water discharged.*

The value of H' in particular cases may be estimated either (*a*) theoretically or (*b*) experimentally.

(*a*) If the loss of energy occurring between the two sections can be estimated theoretically, the value of H' can be computed from its energy meaning as just stated.

(*b*) If the values of H_1 and H_2 can be determined by actual measurement, the value of H' becomes known from the equation $H' = H_1 - H_2$.

In most cases in which the first method is employed the theoretical value of H' involves coefficients whose values can be known only by experiment.

73. Experimental Determination of Lost Head.—The determination of the effective head at any section requires the

measurement of z, p, and v. The elevation above datum may be found by direct measurement or leveling. It remains to consider how pressure and velocity may be determined.

The mean velocity in any section is known from the relation $v=q/F$, as soon as the rate of discharge is known. Methods of measuring q will be discussed in Chapter XIII. If the two cross-sections to be compared are equal, the values of v are equal, and need not be known in order to determine the loss of head between the sections.

The pressure at any section of a pipe may be determined by tapping the pipe and attaching a pressure-gauge. Several forms of pressure-gauges are used.

74. Water Piezometer. — The simplest form of pressure-gauge is a tube inserted in the pipe at right angles to its axis and carried up to such height as may be necessary to prevent overflow, the top being open. Thus, in Fig. 37, the pressure at any point A in the cross-section S is equal to the pressure due to the height of the column of water AA' plus atmospheric pressure; that is, the pressure at any depth h below the top of the column is p_0+wh, this law holding from the top of the column to the bottom of the cross-section S. That it holds throughout the cross-section follows from the fact that the direction of motion of the water is at right angles to the plane of the cross-section. The reasoning of Art. 11 applies if the

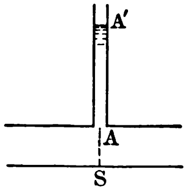


FIG. 37.

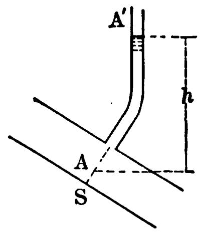


FIG. 38.

points A and B (Fig. 3) are in the plane of the cross-section since the elementary prism AB has no acceleration in the direction of its length.*

* This reasoning is rigorously true on the assumption that all particles

If the pipe is not horizontal (Fig. 38), the law $p=p_0+wh$ still holds throughout the tube and the cross-section S, h being measured vertically.

It obviously makes no difference whether the tube communicates with the cross-section S at the highest point or at some other part of the perimeter. Thus, if Fig. 39 is a cross-sectional view, and if the tubes AA' and BB' are inserted at different points, the tops of the columns (A' and B') will be at the same level.

In order that the piezometer may give a reliable indication of the pressure in the pipe, the axis of the tube at the point

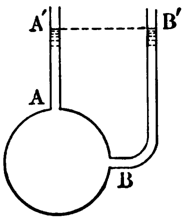


FIG. 39.

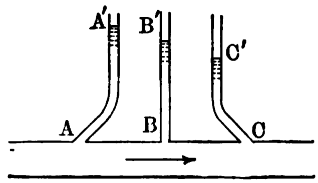


FIG. 40.

of attachment must be at right angles to the axis of the pipe. Thus, if three tubes are inserted, as at A, B, and C (Fig. 40), the friction of the water passing the orifice will increase the pressure in the tube A and decrease it in the tube C, while the column BB' will indicate the pressure correctly. The tops of the columns will stand at different levels, as shown.

The foregoing discussion applies only to the case in which the pressure throughout the cross-section of the pipe is greater than that of the atmosphere.

If, at the point A (Fig. 39), the pressure within the pipe is less than that of the atmosphere, no column of water will

of water move in straight lines parallel to the axis of the pipe, and in straight pipes the conclusion undoubtedly holds within the limits of accuracy ordinarily attainable in hydraulic measurements. For the most accurate work it is best to connect the piezometer tube with a chamber communicating with the cross-section of the pipe by small orifices at several points of the circumference.

be sustained in the tube, but the external pressure will continually force air into the pipe through the tube. The pressure may in such case be measured by what is called a "vacuum piezometer" (Fig. 41). The tube inserted at A is carried upward, then bent down, the end dipping into an open vessel of water at C. The pressure within the bent tube, after a condition of equilibrium has been established, will be less than atmospheric, and water will rise from the vessel to some height CC'. In the other branch of the tube, water will stand at some height AA', which may or may not be zero, depending upon the initial conditions.

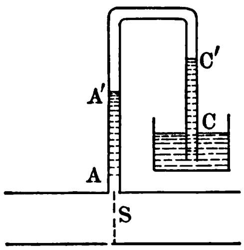


FIG. 41.

Neglecting the weight of the air in the tube, the pressure within the air-space in the tube is uniform. Therefore, at a vertical distance below A' equal to CC' (in the column AA' or the section S), the pressure is atmospheric. The pressure at any point in the section S can therefore be determined if the tops of the two columns AA' and CC' can be observed.

75. Mercury Gauge.—For high values of the pressure the piezometer tube would need to be so long as to render its use impracticable. In such cases it may be possible to use a mercury gauge, such as is represented in Fig. 42.

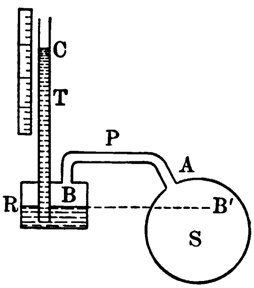


FIG. 42.

The tube P communicates at A with the cross-section of the pipe at which it is desired to measure the pressure, and also communicates with the reservoir R at the top. This reservoir is partly filled with mercury, the tube P and the space in the reservoir above the mercury being filled with water. The vertical tube T, open at both ends, is fitted tightly into the reservoir at the top, and projects far enough down so that the lower end is always immersed in mercury.

Throughout the space above the mercury in the reservoir, the tube P, and the cross-section S of the pipe, the pressure varies according to the hydrostatic law; the pressure at the mercury surface B being equal to that at points at the same level in the cross-section S or in the tube. If the pressure at B is greater than atmospheric, mercury rises in the tube T until the pressure at B due to the mercury column BC, plus atmospheric pressure, is equal to the pressure at the same point communicated through the tube P. If, by means of a fixed vertical scale, the height of the column BC can be measured, the pressure at B, and therefore at any point in the section S, can be determined.

Thus, let h=height of column BC; s=specific gravity of mercury; p_0=atmospheric pressure; p'=pressure at surface of mercury in reservoir. Then

$$\frac{p'}{w}=\frac{p_0}{w}+sh.$$

If p denotes the intensity of pressure at a point in the cross-section S whose vertical distance below the horizontal plane through B is x,

$$\frac{p}{w}=\frac{p'}{w}+x=\frac{p_0}{w}+sh+x.$$

The tube P must be kept free from air, for which purpose an air-cock should be provided at its highest point.

76. Bourdon Gauge.—Where precision and accuracy are not required the form of gauge ordinarily used with steam boilers is often employed for the measurement of water pressure.

EXAMPLES.

1. In order to determine the loss of head between two sections A and B of a straight pipe of uniform cross-section, piezometers were attached at the two sections. The top of the column at A stood 150′ above a horizontal datum plane, and that of the column at B 140′ above the same plane. The center of the pipe at A was 80′, and that at B 60′, above datum.

(*a*) What was the total loss of head between *A* and *B*?

(*b*) What was the pressure at the center of each of the cross-sections *A* and *B*?

(*c*) In which direction was the flow? (How can the direction of flow always be determined from an experiment of this kind?)

2. With the pipe as in Ex. 1, suppose the piezometers to be replaced by mercury gauges. Let the surfaces of mercury in the reservoirs of the two gauges be 82.63′ and 59.80′ respectively above datum, and let the vertical heights of the mercury columns be 12.86′ and 13.46′ respectively.

(*a*) Compute the loss of head between *A* and *B*.

(*b*) Compute the pressure at the center of each of the two sections.

Ans. (*a*) 14.67 ft.

3. Take data as in Ex. 2, except that the size of the pipe is not uniform.

(*a*) What additional data must be given in order that the loss of head may be computed?

(*b*) If the diameter is 12″ at *A* and 10″ at *B*, and the rate of discharge is 6 cu. ft. per sec., what is the loss of head between *A* and *B* when the gauges read as stated in Ex. 2? *Ans.* (*b*) 13.70 ft.

77. Loss of Head in Standard Orifice.—The velocity of efflux from a standard sharp-edged orifice (Art. 45) is

$$v = c'\sqrt{2gh},$$

in which the coefficient of velocity c' has a value fairly well established by experiment.

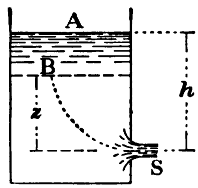


FIG. 43.

Comparing the values of the effective head at two points, one of which is within the reservoir where the velocity is practically zero, and the other in the contracted section of the jet (as the points *B* and *S*, Fig. 43), we have

$$H_1 = h + \frac{p_0}{w},$$

$$H_2 = \frac{v^2}{2g} + \frac{p_0}{w};$$

p_0 being atmospheric pressure and v the velocity of flow at section *S*, and the datum plane being taken at the level of the

center of the orifice. Hence

$$H' = H_1 - H_2 = h - \frac{v^2}{2g} = (1 - c'^2)h = \frac{1 - c'^2}{c'^2} \cdot \frac{v^2}{2g} = \left(\frac{1}{c'^2} - 1\right)\frac{v^2}{2g}.$$

The value of H' is thus expressed in terms either of the head on the orifice or of the velocity of the jet.

If $c' = 0.98$ (Art. 45), the values become

$$H' = 0.040h = 0.041\frac{v^2}{2g}.$$

78. Loss of Head in Short Tube.—Consider next the loss of head in case of discharge from a short cylindrical tube. The length is assumed to be just sufficient so that the stream, after converging as it enters the tube, expands and fills the tube at the outer end (Art. 38). The inner edge of the tube is supposed to be square (Fig. 44).

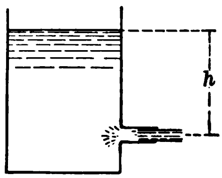


FIG. 44.

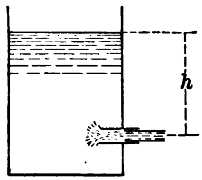


FIG. 45.

The expression for H' in terms of the coefficient of velocity is the same as in the case of the orifice; but the value of this coefficient is different: As determined experimentally it is

$$c' = 0.82 \text{ (nearly)},$$

which gives

$$H' = 0.33h = 0.49\frac{v^2}{2g}.$$

If the tube projects within the vessel, as in Fig. 45, the velocity of the jet is less than in the preceding case. Experiment gives

$$c' = 0.72 \text{ (about)}$$

and

$$H' = 0.48h = 0.93\frac{v^2}{2g}.$$

79. Loss of Head in Straight Pipe of Uniform Cross-section —The loss of head in a given length of a uniform straight pipe depends upon the nature of the inner surface, the diameter, and the velocity of flow.* Experiment indicates that it is independent of the pressure existing in the pipe. Since loss of head is proportional to energy dissipated (Art. 63), it obviously depends upon the friction between the water and the pipe, as well as upon the friction and impact of the particles of water among themselves.

No theoretical formula has been proposed which can be depended upon to give more than a rough approximation to the actual losses of head in pipes. The following formula, applicable to a uniform pipe of any form of cross-section, is often employed, and is the basis of most of the formulas that will be considered in the following pages:

Let F = cross-sectional area;

C = length of perimeter of cross-section;

$r = \frac{F}{C}$;

H' = loss of head in length l of pipe;

v = mean velocity of flow.

Then
$$H' = f'\frac{l}{r}\frac{v^2}{2g}.$$

In this formula f' is a coefficient whose value depends upon the roughness of the surface of the pipe, and also (although to a less degree) upon the size of the pipe and the velocity of flow. The quantity r is a length, and is called the *hydraulic radius* of the cross-section; its value depending only upon the form and size of the section.

* Doubtless, also, upon the temperature of the water; but regarding this little experimental knowledge is available.

The theoretical basis of this formula will be considered in Chapter IX.

80. Cylindrical Pipe.—If the cross-section of the pipe is a circle of diameter d, we have

$$C=\pi d; \quad F=\frac{\pi d^2}{4}; \quad r=\frac{F}{C}=\frac{d}{4}.$$

Hence (writing f for $4f'$) the formula becomes

$$H'=f\,\frac{l}{d}\,\frac{v^2}{2g}.$$

Experiment indicates that the coefficient or "friction factor" f in this formula is not constant for a given kind of pipe but varies with both d and v, decreasing as each of these quantities increases. The variation with v appears, however, to be less important than that with d. For new or clean cast-iron pipe, and for velocities within the range ordinarily occurring in practice, Darcy recommended a formula equivalent to the following (d being in feet):

$$f=.0199+\frac{.00166}{d}.$$

For old pipe these values should be doubled.*

This formula may be used in solving the examples which follow. The values of f for pipes of different kinds will be considered further in Chapter IX.

81. Loss of Head at Entrance to Pipe.—If water is conducted from a reservoir by a straight pipe, the conditions of flow are less uniform near the entrance to the pipe than they become a little farther on. The contraction and expansion of the entering stream result in a loss of head near the entrance

* Recherches expérimentales relatives au mouvement de l'eau dans les tuyeaux. Chapter IV, also p. 228.

that is much greater than occurs in an equal length where the flow has become uniform. The entrance loss is, however, small in comparison with the total loss unless the pipe is short, and for very long pipes it may be neglected.

For those cases in which entrance loss is important enough to be considered, a formula for it is obtained by assuming that the conditions in the entrance portion of the pipe are similar to those existing in a short tube discharging into the atmosphere. It is assumed that, for any given pipe, the loss of head at entrance varies mainly with the velocity of flow, and that it has the same value as for a short tube of the same diameter, provided the conditions are such that the velocities are equal in the tube and the pipe. The entrance loss may therefore be expressed in terms of the velocity by the formulas already given (Art. 78) for the case of a short tube. Practically, the following may be taken as sufficiently correct:

$$H' = .5\frac{v^2}{2g} \text{ for pipe not projecting into reservoir;}$$

$$H' = \frac{v^2}{2g} \text{ for projecting pipe.}$$

82. Loss of Head Due to Sudden Enlargement of Pipe.—If the cross-section of a pipe enlarges suddenly from an area F_1 to an area F_2 (Fig. 46), the irregular motions of the particles of water due to the sudden expansion of the stream cause a dissipation of energy and consequent loss of head. If v_1 and v_2 denote the values of the velocity at the two sections, the loss of head is found by experiment to be given approximately by the formula

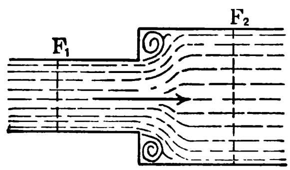


FIG. 46.

$$H' = \frac{(v_1 - v_2)^2}{2g}.$$

Since $F_1v_1 = F_2v_2$, this formula may be written

$$H' = \left(1 - \frac{F_1}{F_2}\right)^2 \frac{v_1^2}{2g} = \left(\frac{F_2}{F_1} - 1\right)^2 \frac{v_2^2}{2g}.$$

Special Case.—If the ratio F_1/F_2 is very small, the above formula reduces to

$$H' = \frac{v_1^2}{2g}.$$

This applies to the case in which a pipe discharges into a reservoir at a point below the water surface.

83. Loss of Head Due to Sudden Contraction of Stream.—If the cross-section of the pipe decreases suddenly in the direction of the flow (Fig. 47), there is a loss of head, but less than in the case of sudden enlargement. The value of the loss depends upon the ratio of the two cross-sections in a way that can be known only by experiment. Let F_1 and F_2 denote the larger and smaller cross-sections respectively, v_1 and v_2 being the corresponding velocities; then the loss of head may be expressed by the formula

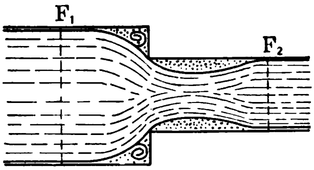


FIG. 47.

$$H' = k\frac{v_2^2}{2g},$$

k being an experimental coefficient depending upon F_2/F_1. The following values of k are based upon data given by Weisbach:

$\frac{F_2}{F_1}$	.10	.20	.30	.40	.50	.60	.70	.80	.90	1.00
k	.362	.338	.308	.267	.221	.164	.105	.053	.015	0

84. Loss of Head Due to Obstruction in Pipe.—If the cross-section of a pipe at any point is partly closed by an obstruction, it may be assumed that the loss of head is due mainly to the

sudden expansion of the stream just beyond the obstruction, and the loss may be estimated as in the case of a sudden enlargement in a pipe (Art. 82). Thus, let

F = cross-section of pipe;
F' = area at obstructed section;
v = velocity where cross-section is F;
v' = " " " " F'.

The formula becomes

$$H' = \frac{(v'-v)^2}{2g} = \left(\frac{F}{F'}-1\right)^2 \frac{v^2}{2g}.$$

85. Loss of Head Caused by Bend.—The loss of head in a curved pipe is greater than that in an equal length of straight pipe. No rational formula can be given for estimating its value, and the empirical rules commonly given are probably not very reliable. It is reasonable to suppose that for pipe of a given size the loss per unit length is greater as the radius of the curve is less, and that the total loss due to a curve of given radius increases with the total angle of deflection, i.e., with the length of the curve. For curves of short radius, having the same total angle of deflection but different radii, the usual assumption is that the loss of head increases as the ratio of the radius of the curve to that of the pipe decreases. For larger values of the radius, however, the total loss for a curve of given total deflection increases with the radius because of the increased length of the curve.

For 90° curves of short radius, the formula usually given is that of Weisbach, who assumed

$$H' = k\frac{v^2}{2g},$$

and gave for k the empirical formula

$$k = 0.131 + 1.847\left(\frac{r}{R}\right)^{\frac{7}{2}},$$

in which r is the radius of the pipe and R that of the bend.

That for a given value of R the loss of head should increase with the diameter of the pipe seems hardly reasonable, even when R is small. At all events, Weisbach's formula should not be used when the radius of the curve is greater than four or five times the diameter of the pipe.

For curves of longer radius experimental data are too limited to form the basis of any general formula.*

86. Hydraulic Gradient.—If piezometers be inserted into a pipe at various points of its length, the line joining the highest points of all the piezometric columns is called the *hydraulic gradient*. In Fig. 48, $A'B'$ represents the hydraulic gradient for the pipe AB.

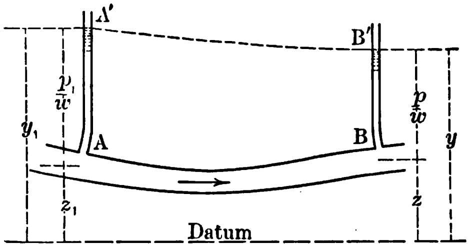


FIG. 48.

The general nature of the curve of the hydraulic gradient may be inferred from the equation of energy. Thus, in Fig. 48, let z_1, v_1, p_1 refer to some given section A, and z, v, p to any section B, down-stream from A. Let y denote the elevation above datum of the top of the piezometric column at any section, y_1 being the value of y at A. The equation of energy

* Experiments on the loss of head caused by 90° bends in pipes of 12″, 16″, and 30″ diameter, made by Gardner S. Williams, Clarence W. Hubbell and George H. Fenkell, are described in Vol. XLVII of the Transactions of the American Society of Civil Engineers. The loss of head was in each case measured for a total length of pipe which included two tangents connected by the bend, and the effect of the bend was estimated by comparison with the loss in an equal total length of straight pipe. Those of the results showing the greatest regularity indicated that the total loss of head due to each bend in the 30″ pipe was approximately three times the loss caused by an equal length of straight pipe.

gives

$$z+\frac{p}{w}=z_1+\frac{p_1}{w}+\frac{v_1^2}{2g}-\frac{v^2}{2g}-H',$$

if H' is the head lost between A and B. Since $y=z+p/w$, the equation may be written

$$y=y_1+\frac{v_1^2}{2g}-\frac{v^2}{2g}-H'.$$

Consider the following special cases:

(1) Suppose the cross-section of the pipe is constant. Then $v=v_1$ and

$$y=y_1-H'.$$

That is, the hydraulic gradient falls, between any two sections, by just the amount of the head lost between those sections.

On the assumption of no loss of head, the hydraulic gradient, in case of uniform cross-section, would be a horizontal line.

In a straight pipe of uniform interior roughness the loss of head is proportional to the length, and hence the hydraulic gradient has a uniform downward slope in the direction of flow.

A bend or obstruction would cause a sudden drop in the hydraulic gradient.

(2) Suppose the cross-section of the pipe variable. Then, in addition to the effect of loss of head, the hydraulic gradient will rise or fall with varying velocity; rising as the velocity decreases and falling as it increases.

87. Hydraulic Slope.—The fall of the hydraulic gradient per unit length of the pipe is called the *hydraulic slope.*

If l denotes the length of the pipe measured from a fixed section (as A, Fig. 48), and s the hydraulic slope at the point considered, y being the ordinate of the hydraulic gradient at that point (as B, Fig. 48), we have

$$s=-\frac{dy}{dl}.$$

If the hydraulic slope is constant from A to B,

$$s=\frac{y_1-y}{l}.$$

If the cross-section is constant, so that the fall of the hydraulic gradient is wholly due to loss of head (as in case (1), Art. 86), $y_1-y=H'$ and

$$s=\frac{H'}{l}.$$

88. Applications of Theory.—In the preceding chapter was treated the method of applying the general equation of energy to the solution of problems in steady flow, neglecting losses of energy by dissipation. The general method there outlined may readily be extended so as to take account of losses of head, the value of the term H' in the equation of energy being expressed in accordance with the foregoing principles.* This will be illustrated by the solution of an example.

EXAMPLES.

1. The pipe AB (Fig. 49) is 1000′ long and 6″ in diameter, and the length AC is 550′. Compute (*a*) the rate of discharge; (*b*) the pressure at the center of each of the cross-sections A, B, C.

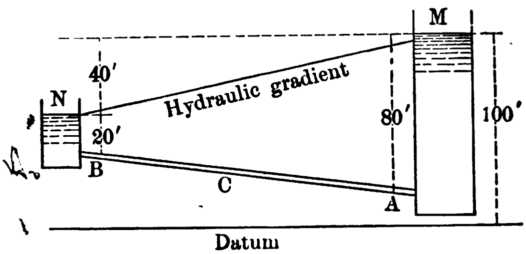


FIG. 49.

Solution.—(*a*) First apply the equation of energy, comparing the points M and N, at the surfaces of the two reservoirs; H_1 denoting the

* In solving the following examples, the coefficient f, for estimating the frictional loss of head in pipes, may be determined by Darcy's formula (Art. 80).

effective head at M, and H_2 that at N, while H' denotes the total head lost from M to N.

Taking datum plane as shown in the figure,

$$H_1 = 100, \quad H_2 = 60.$$

The value of H' is made up of three parts, each of which may be expressed in terms of the velocity of flow in the pipe. Let v denote this velocity; then we have:

(1) The loss at entrance to the pipe (Art. 81) may be taken as

$$0.5\frac{v^2}{2g}.$$

(2) The loss between A and B due to friction (Art. 80) is

$$f\frac{l}{d}\frac{v^2}{2g} = 2000f\frac{v^2}{2g}.$$

Darcy's formula gives $f = .0232$, which reduces the expression for the frictional loss to

$$46.4\frac{v^2}{2g}.$$

(3) The loss of head due to the sudden destruction of the velocity at the point of discharge into the reservoir N is (by Art. 82, putting $F_1/F_2 = 0$)

$$\frac{v^2}{2g}.$$

Combining the three losses,

$$H' = 47.9\frac{v^2}{2g},$$

and the equation $H_1 - H_2 = H'$ becomes

$$100 - 60 = 47.9\frac{v^2}{2g},$$

from which

$$\frac{v^2}{2g} = 0.833; \quad v = 7.32 \text{ ft. per sec.}$$

The rate of discharge is

$$q = Fv = 0.196 \times 7.32 = 1.43 \text{ cu. ft. per sec.}$$

(b) Applying the energy equation to sections M and A (the latter

being a short distance within the pipe), the value of H' is $0.5(v^2/2g) = 0.42$. Also, with datum as before,

$$H_1 = 100, \quad H_2 = 20 + \frac{p_2}{w} + 0.83,$$

and the equation $H_1 - H_2 = H'$ gives

$$\frac{p_2}{w} = 78.7 \text{ ft.} = \text{pressure head at } A.$$

The absolute pressure at A is $p_2 + p_0$, since in this solution atmospheric pressure has been called zero.

By a similar method the pressure head at C is found to be 46.3 ft.

The hydraulic gradient is in this case a straight line except near the entrance to the pipe. Starting at the surface of the reservoir M, it falls by an amount $1.5(v^2/2g) = 1.25$ ft. by reason of the entrance loss and the velocity head, then slopes uniformly by reason of the frictional loss, falling .03875 ft. per foot of length of the pipe until the surface of the reservoir N is reached.

2. In Fig. 49, how large must the pipe be in order that the rate of discharge may be 5 cu. ft. per sec.? In that case, what pressure exists in the pipe at C?

Ans. $d = .82$ ft. Pressure head at $C = 46.4$ ft. above atmosphere.

3. A pipe AB, 5000′ long, is to discharge 5 cu. ft. per sec. in the direction AB. The point B is 20′ higher than A, and it is required that the pressure head at A is to be 200′, and that at B 100′. What must be the diameter of the pipe? *Ans.* Nearly 1 foot.

4. A pipe AB is 6000′ long and 8″ in diameter, the point B being 80′ higher than A. If the pressure head is 100′ at B and 190′ at A, in which direction is the flow, and what quantity is discharged per sec.?

Ans. $q = .623$ cu. ft. per sec. in direction AB.

5. In Ex. 4, if the discharge is to be 3 cu. ft. per sec. in the direction AB, what must be the excess of pressure at A over that at B?

Ans. 311 ft. head.

89. Loss of Head in Capillary Tubes.—Experiment indicates that the loss of head in tubes of very small diameter varies according to quite different laws from those applying to ordinary pipes. Poiseuille gave a formula equivalent to the following, deduced from experiments on glass tubes from .014 mm. to .65 mm. in diameter:

$$H' = k\frac{lv}{d^2}.$$

The coefficient k depends upon the viscosity of the water, and decreases as the temperature increases.

90. Flow Through Gravel.—When water flows through porous earth or gravel, the velocity is in general small, and the loss of head is found to vary approximately as the first power of the velocity. The loss per unit distance along the stream depends upon the character of the material through which the water is flowing, being greater as this is finer and more compact.

91. Flow of Wells.—A bed of gravel or other porous material lying between two layers of clay or impervious rock may carry water under pressure. Such a case is represented in Fig. 50,

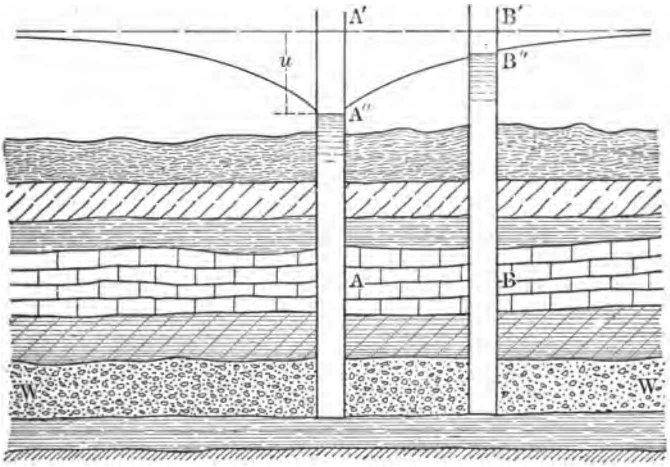


FIG. 50.

which shows two tubular wells A and B which have been driven from the surface of the ground down to and through the water-bearing stratum WW. The portion of each tube within the porous bed is perforated so as to permit free entrance of water. If the same stratum is tapped by several such tubes, the water will rise in all to the same level *if the water in the stratum is at rest.* If, however, as is generally the case, the water is flowing through the porous layer, different pressure columns will stand at different levels.

Considering a case in which the water is at rest, let the two tubes A, B (Fig. 50) be carried high enough to prevent overflow, and let $A'B'$ be the horizontal plane to which the water rises. Now suppose water to be drawn from the tube A at a uniform rate, either by pumping or by cutting the tube off at some distance below A' so as to permit overflow. Flow at once begins into the tube and toward it from all directions through the gravel, and a steady condition ensues in which the quantity entering the tube in a given time is equal to the quantity flowing out. In this steady condition there is a definite relation between the rate of discharge and the drop of the water surface below the plane $A'B'$. The nature of this relation depends upon the way in which loss of head varies with velocity of flow both in the porous stratum and in the tube.

92. Relation between Rate of Discharge and Fall of Water Surface.—Assuming the horizontal plane $A'B'$ as datum, let the equation $H_1 - H_2 = H'$ be applied, taking the up-stream point at a great distance from the well and the down-stream point at the water surface in the tube.

Let v = velocity of flow in the tube;
q = rate of discharge;
u = drop of water surface $= A'A''$.

Then
$$H_1 = 0, \quad H_2 = -u + \frac{v^2}{2g}.$$

To express H' in terms of v we assume the law stated in Art. 90, that the loss of head due to flow through a porous stratum varies directly as the velocity, other conditions remaining constant. Although the velocity has different values at different points, the values at all points may be assumed to vary directly as v, so that the total loss of head in the gravel may be expressed by a term c_1v, c_1 being a constant. The loss in the tube is doubtless ordinarily small in comparison with that in the gravel, but to take account of this and the loss in entering the tube a term cv^2 may be introduced. The total loss of

head thus takes the form

$$H' = c_1 v + cv^2,$$

and the equation $H_1 - H_2 = H'$ becomes

$$u - \frac{v^2}{2g} = c_1 v + cv^2,$$

from which

$$u = c_1 v + c_2 v^2,$$

c_1 and c_2 being constants. Since v varies directly as q,

$$u = k_1 q + k_2 q^2,$$

k_1 and k_2 being constants whose values for any given well must be determined by experiment.

The simpler equation

$$u = k_1 q$$

is often sufficiently correct.

These results are verified by experiments on the actual discharge of wells for different heights of overflow.

93. Hydraulic Gradient in Case of Flowing Well.—Since the hydraulic gradient falls in the direction of flow by the amount of the loss of head (the change in the velocity head being in this case inappreciable), the gradient is a surface which rises in all directions in going from the well, approaching tangency with the horizontal surface $A'B'$. The position of the water surface in the tube B indicates the height of the hydraulic gradient at that point. The difference of level of the water surfaces in the two tubes (corrected slightly for the velocity head at A'' and the loss of head due to the tube) is the loss in the gravel from B to A. The less the porosity of the gravel the greater is the value of this loss, i.e., the rate at which the hydraulic gradient rises in passing away from the well is greater as the stratum is less pervious.

If water be drawn from both wells at the same time, the drop in the surface of each will be due in part to its own discharge and in part to that of the other well. The quantity of water that can be drawn from a well with a given depression of its surface below the static level $A'B'$ (Fig. 50) may thus be influenced in an important manner by the flow of other wells in the vicinity.

CHAPTER VII.

GENERAL EQUATION OF ENERGY WHEN PUMP OR MOTOR IS USED.

94. Equation of Energy for Stream Flowing Through Motor. —In deducing the general equation of energy (Art. 60), it was assumed that no mechanical energy is imparted to the water between the two sections A and B (Fig. 25), and that no energy is taken from it except such as is dissipated by reason of friction between the particles of water and the pipe, and friction and impact among the particles of water themselves.

If a motor is driven by the stream, mechanical energy is taken by the motor from the water, and this must be taken into account in forming the equation of energy. This case may be represented by Fig. 51, in which M is a motor through which water flows in the direction AB.

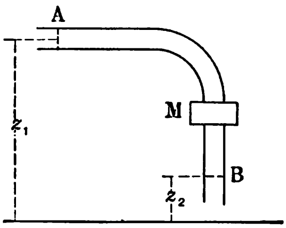


Fig. 51.

Let H' now mean the total mechanical energy lost by the water between A and B, per pound of water discharged, and let

$$H' = h' + h'', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which h' is the part of H' corresponding to energy dissipated and h'' the part corresponding to energy received by the motor. Then the energy gained by the volume AB per unit time is

$$W\left(z_1 + \frac{p_1}{w} + \frac{v_1^2}{2g}\right) = WH_1,$$

and the energy lost is

$$W\left(z_2+\frac{p_2}{w}+\frac{v_2^2}{2g}\right)+Wh'+Wh''=W(H_2+h'+h'').$$

Equating,

$$H_1=H_2+h'+h'',$$

or

$$h''=H_1-H_2-h'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

95. Formula for Energy Transferred to Motor.—The amount of energy received by the motor from the stream per unit time is

$$Wh''=W(H_1-H_2-h'), \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and the horse-power imparted to the motor is

$$\text{H.P.}=\frac{W}{550}(H_1-H_2-h'), \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

W being in pounds per second and H_1, H_2, h' in feet.

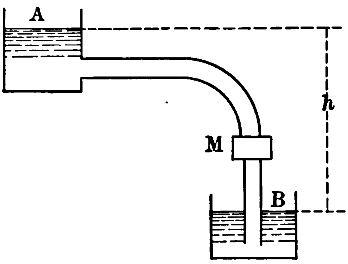


FIG. 52.

Suppose the stream passing through the motor flows from one reservoir to another, the surface of the discharge reservoir being h ft. lower than that of the supply reservoir (Fig. 52). Taking sections A and B at the reservoir surfaces, we have

$$H_1-H_2=h.$$

Hence in this case we may write

$$Wh''=W(h-h'), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Wh' being the energy dissipated per second in the entire stream from A to B. The effect of the frictional losses of energy is thus to decrease by Wh' the energy imparted to the motor per second, which amounts practically to decreasing the available fall of the water by h'.

If the motor discharges directly into the air, equations (3) and (4) still apply, but instead of reducing this to the form (5), it is more convenient to change the notation. If h now denotes the fall from surface of supply reservoir to point of outflow from motor, and v_2 the velocity of outflow,

$$H_1 - H_2 = h - \frac{v_2^2}{2g},$$

and (3) becomes

$$Wh'' = W\left(h - h' - \frac{v_2^2}{2g}\right). \quad . \quad . \quad . \quad . \quad . \quad (6)$$

96. Equation of Energy for Stream Flowing Through Pump. If the flow of a stream is maintained or aided by a pump, the mechanical energy given up by the pump to the water must be taken into account in forming the equation of energy. Let this case be represented by Fig. 53, P representing the pump, and AB being the direction of flow.

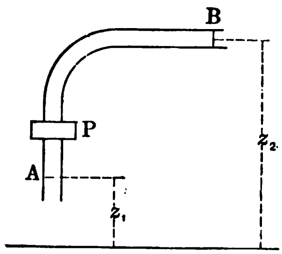


FIG. 53.

Let h''' denote the energy received by the water from the pump per pound of water discharged, and let h' denote as above the energy lost by dissipation between A and B per pound of the discharge. Then H' being the net loss of head from A to B,

$$H' = h' - h'''. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Reasoning as in Art. 60, we may write at once

$$H_1 - H_2 = H' = h' - h''',$$

from which

$$h''' = H_2 - H_1 + h'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

It is seen that the action of the pump causes the effective head to increase in the direction of flow by the amount h'''. If h''' is greater than h', H_2 is greater than H_1; i.e., the total loss of head H' is negative.

97. Formula for Energy Used in Pumping.—The total mechanical energy supplied by the pump per second is

$$Wh''' = W(H_2 - H_1 + h'). \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Here h' must be understood to include energy lost by hydraulic friction within the pump as well as in other parts of the stream.

The rate of working of the pump, in horse-power, is

$$\text{H.P.} = \frac{Wh'''}{550} = \frac{W}{550}(H_2 - H_1 + h'). \quad . \quad . \quad . \quad (10)$$

Suppose water is pumped from one reservoir to another (Fig. 54), the vertical lift between reservoir surfaces being h. Taking the sections A and B at the two surfaces, we have

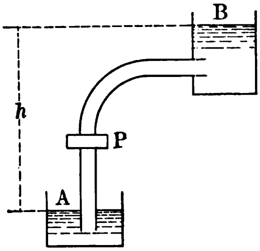


FIG. 54.

$$H_2 - H_1 = h,$$

and (9) may be written in the form

$$Wh''' = W(h + h').$$

The work done in pumping is thus the same as would be done in lifting the water a height $h + h'$ against gravity if all waste of energy could be avoided.

EXAMPLES.

1. Water is supplied to the motor M (Fig. 55) from the head-race A and conducted from M to the tail-race D. The supply-pipe and waste-pipe are each 6″ in diameter. Suppose that no energy is lost by dissipation in any part of the apparatus, and that the discharge is 3 cu. ft. per sec. Compute the pressure at B and at C, and the H.P. transmitted to the motor.

Ans. Pressure head at $B = p_0/w + 6.4'$; pressure head at $C = p_0/w - 17.6'$; H.P. $= 15$.

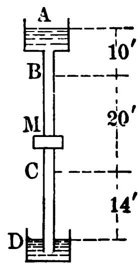


FIG. 55.

2. Water is pumped from one reservoir into another through 1000′ of 6″ pipe. The surface of the second reservoir is 20′ higher than that of the first. If the quantity delivered is 3 cu. ft. per sec., (*a*) at what rate is energy imparted to the water by the pump? (*b*) What fraction of this energy is utilized in lifting the water? (*c*) The losses

of energy in the flow are equivalent to what added lift? (*d*) Draw hydraulic gradient.

Ans. (*a*) 66.8 H.P. (*b*) About 10%. (*c*) 176 ft.

3. Water flows from one reservoir into another, the surface of the second being 10′ lower than that of the first. The flow takes place through a pipe 8″ in diameter and 1600′ long, the intake end projecting into the reservoir, while the other end discharges below the water surface. (*a*) If the flow is due to gravity alone, what quantity is discharged per second? (*b*) In order to double the rate of discharge, at what rate must energy be given to the water by the pump?

Ans. (*a*) 1.19 cu. ft. per sec. (*b*) 8.12 H.P.

4. In Fig. 56, *C* is the cylinder of a piston-pump which is lifting water from the reservoir *B* and discharging it at *D*. The diameter of the pipe is everywhere 2″, and the radius of the bend is 3″.

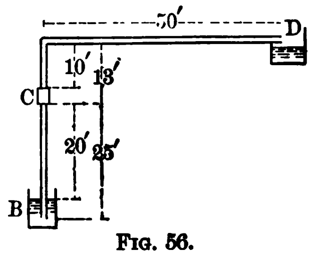


FIG. 56.

(*a*) At what maximum rate can water be pumped without causing a pressure head of less than 4′ (absolute) at any point in the pipe? Where will the minimum pressure occur?

(*b*) Compare the pressures just below and just above the pump cylinder.

(*c*) Compute the H.P. of the pump.

[Estimate the loss due to a bend by formula of Weisbach. Solve, also, disregarding this loss and see whether it is important.]

Ans. (*a*) 0.218 cu. ft. per sec. (*b*) Below, $p/w = 4'$ (absolute); above, $p/w = 26.9'$ above atmosphere. (*c*) 1.48 H.P. The pressure head above the cylinder is increased 0.26 ft. by the bend, and the H.P. is increased about 0.44%. [These results are based on the assumption that $p_0/w = 34'$.]

CHAPTER VIII.

FLOW IN PIPES: SPECIAL CASES.

98. Flow From Reservoir Through Pipe.—If a pipe leads from a reservoir and discharges into the atmosphere, formulas for the velocity of flow and the rate of discharge may be obtained by applying the general equation of energy $H_1 - H_2 = H'$.

Referring to Fig. 57, let

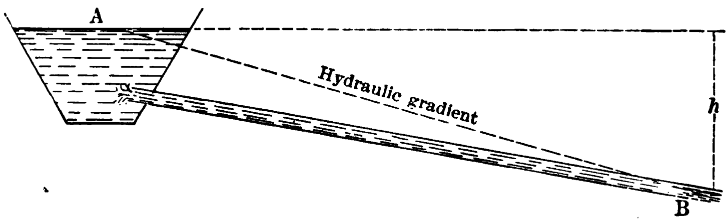


FIG. 57.

h = total fall from reservoir to center of outflowing stream;
v = mean velocity in pipe;
v' = " " of outflowing stream;
l = length of pipe;
d = diameter of pipe.

Writing the equation of energy for the stream AB, taking datum plane through the center of the stream at B,

$$H_1 = h = \text{effective head at } A;$$

$$H_2 = \frac{v'^2}{2g} = \text{effective head at } B;$$

$$H_1 - H_2 = h - \frac{v'^2}{2g}.$$

If the pipe is of uniform section, straight and unobstructed, H' is the sum of two parts,—the loss at entrance, and the loss by friction in the pipe. If there are bends or obstructions of any kind, or expansions or contractions of the cross-section, these cause additional losses which may be taken as approximately proportional to the square of the velocity.

CASE I. Let the cross-section of the outflowing stream or jet be equal to that of the pipe, so that $v' = v$; then

$$H_1 - H_2 = h - \frac{v^2}{2g}.$$

The losses of head may be expressed as follows:

The entrance loss is $m\frac{v^2}{2g}$, where $m=1$ for projecting and 0.5 for non-projecting pipe (Art. 81).

The frictional loss in the pipe is $f\frac{l}{d}\frac{v^2}{2g}$, in which f has the meaning explained in Art. 80.

The loss due to bends or obstructions is expressed by a term $n\frac{v^2}{2g}$, the value of n depending upon the circumstances of each particular case.

Combining all losses,

$$H' = \left(m + f\frac{l}{d} + n\right)\frac{v^2}{2g}.$$

The equation $H_1 - H_2 = H'$ now becomes

$$h - \frac{v^2}{2g} = \left(m + f\frac{l}{d} + n\right)\frac{v^2}{2g}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The method of solving this equation in particular cases will be illustrated below.

CASE II. Let the cross-section of the jet be less than that of the pipe.

If F = cross-section of pipe and F' = that of jet, let $F = cF'$. Then $v' = cv$ and

$$H_1 - H_2 = h - \frac{(cv)^2}{2g}.$$

To the losses of head occurring in Case I must be added a loss due to the contraction of the stream at the point of outflow. This loss may be expressed by a term $k\frac{v'^2}{2g}$, the value of k depending upon the ratio of contraction and the construction of the pipe and orifice. Combining all losses,

$$H'=\left(m+n+f\frac{l}{d}\right)\frac{v^2}{2g}+k\frac{(cv)^2}{2g},$$

and the equation of energy becomes

$$h-\frac{(cv)^2}{2g}=\left(m+n+f\frac{l}{d}\right)\frac{v^2}{2g}+k\frac{(cv)^2}{2g}. \quad . \quad . \quad . \quad (2)$$

Equation (1) is a special case of (2), in which $c=1$, $k=0$. Equation (2) may be written in the form

$$h=\left[(1+k)c^2+m+n+f\frac{l}{d}\right]\frac{v^2}{2g}. \quad . \quad . \quad . \quad . \quad (3)$$

From it may be computed any one of the quantities v, h, d, if the others are known. For certain applications it is convenient to introduce the rate of discharge q instead of v. Thus, suppose it is required to estimate the size of the pipe which will give a certain rate of discharge, the total fall and length of pipe being known.

Since $v=q/F=4q/\pi d^2$, equation (3) may be written

$$h=\frac{8q^2}{\pi^2 gd^4}\left[(1+k)c^2+m+n+f\frac{l}{d}\right],$$

from which

$$d^5=\frac{8q^2}{\pi^2 gh}\left([(1+k)c^2+m+n]d+fl\right), \quad . \quad . \quad . \quad . \quad (4)$$

or

$$d^5=Ad+B, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4')$$

A and B being constants.

In many cases, especially for long pipes discharging as in Case I, the term Ad is small in comparison with B, and the solution may be made by successive approximations.

As an example, let $h=50'$, $l=2500'$, $q=10$ cu. ft. per sec., $c=1$, $k=0$, $m=1$, $n=0$. These values reduce equation (4) to

$$d^5=.0504(2d+2500f).$$

For a first approximate solution assume $f=.02$ and neglect the term $2d$, which is small in comparison with $2500f$. This gives $d=1.20$, and the value of f given by Darcy's formula is .0213. Using these results in the above equation,

$$d^5=.0504(2.40+53.25)=2.81,$$
$$d=1.23'=\text{nearly } 15''.$$

If the term Ad is not small in comparison with B, solution by trial is less easy, but is always possible.

EXAMPLES.

1. Let the diameter of the pipe be 6″, the length 850′, and the fall to the point of outflow 40′. Assume that the pipe projects into the reservoir, and that the discharge takes place through a nozzle 2″ in diameter, causing a loss of head $0.1(v'^2/2g)$, v' being the velocity of the jet. (a) Compute the rate of discharge. (b) Draw the hydraulic gradient.
Ans. (a) 0.88 cu. ft. per sec.

2. In Ex. 1, let the diameter of the pipe be 12″, the remaining data being unchanged. Compute q, and draw the hydraulic gradient.
Ans. $q=1.05$ cu. ft. per sec.

3. In Ex. 1, compute q on the assumption that the cross-section of the jet equals that of the pipe. Draw the hydraulic gradient.
Ans. $q=1.55$ cu. ft. per sec.

4. Let $h=40'$, $l=850'$, $q=16$ cu. ft. per sec. Assuming no contraction of the stream at outlet, and taking the coefficient of entrance loss as 0.5, compute the size of the pipe. Draw the hydraulic gradient.
Ans. Diameter = about 15″.

5. Water is conducted from a reservoir to a point 300′ lower than the reservoir surface. The conduit is a 4″ pipe, 1200′ long, projecting into the reservoir. Discharge occurs into the atmosphere through a nozzle, delivering a stream 1″ in diameter, the coefficient of velocity for

the nozzle being 0.9. (*a*) Compute the rate of discharge. (*b*) Compute the available energy of the jet per second. (*c*) If the jet drives a motor with an efficiency of 0.75, what H.P. is realized? (*d*) What fraction of the total energy due to the fall is utilized?

[The loss of head in the nozzle may be expressed in terms of the velocity of the jet and the coefficient of velocity in the same way as in the case of a short tube (Art. 78).]

Ans. (*a*) 0.602 cu. ft. per sec. (*c*) 12.9 H.P. (*d*) 63%.

6. Taking data as in example 5, and assuming the supply of water to be unlimited, what sized nozzle will deliver the greatest amount of energy to the motor in a given time? [Assume the coefficient of velocity of the nozzle to be 0.9 for all cases.]

Ans. Diam. of nozzle = 0.287 × diam. of pipe.

99. Pressure at a Given Point to Have an Assigned Value.— Let it be required to conduct water from a reservoir to a certain point, the rate of discharge being given, and the pressure within the pipe at the point of delivery having a specified value. Thus, in Fig. 58, let B be a point at which water is to be delivered, the pressure head there being required to have the value y.

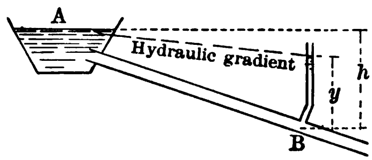


FIG. 58.

Let l = length of pipe to B, d = diameter, h = depth of B below reservoir surface.

Taking datum plane through the center of the pipe at B, the values of the effective head at A and B are

$$H_1 = h, \quad H_2 = y + \frac{v^2}{2g};$$

while the loss of head between A and B is

$$H' = \left(m + n + f\frac{l}{d}\right)\frac{v^2}{2g}.$$

The equation of energy therefore becomes

$$h - y - \frac{v^2}{2g} = \left(m + n + f\frac{l}{d}\right)\frac{v^2}{2g}. \quad . \quad . \quad . \quad . \quad (5)$$

This is identical with equation (1) with $h - y$ substituted for h.

EXAMPLE.

Water is to be delivered through a pipe 1800′ long to a point 70′ below the level of the reservoir surface. The rate of discharge is to be 15 cu. ft. per sec., and the pressure head at the point of delivery 40′. What should be the size of the pipe? *Ans.* $d = 18''$.

100. Branching Pipe.—If water is delivered through a main pipe with branches, the equation of energy must be applied to each portion of the pipe separately, sufficient data being known or assumed to make the problem of flow determinate.

Thus, as a simple case, consider the main pipe AB with two branches BC, BD (Fig. 59), and let the given data be the

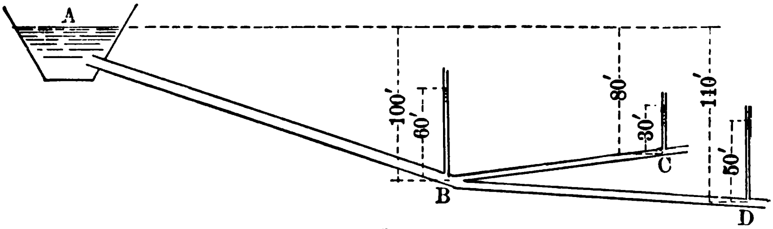


FIG. 59.

rate of discharge at C and at D, and the pressure at each of the three points B, C, D. The method of treatment may be illustrated by a numerical case.

Let 8 cu. ft. per sec. be delivered at C and 10 cu. ft. per sec. at D; let the depth of B below the reservoir surface be 100′, that of C 80′, and that of D 110′; let the length of the main pipe to B be 2000′, that of BC 1200′, and that of BD 1600′; and let the pressure head have the value 60′ at B, 30′ at C, and 50′ at D.

Let d, d', d'' be the diameters of the three pipes AB, BC, CD; and v, v', v'' the corresponding velocities of flow.

The losses of head at entrance to the branch pipes cannot be accurately estimated, but will be small in comparison with the frictional losses in the pipes and will be neglected.

For the pipe AB the equation is

$$40 - \frac{v^2}{2g} = \left(1 + f\frac{2000}{d}\right)\frac{v^2}{2g},$$

from which, since $v=\frac{18}{\pi d^2/4}$, the value of d may be found as in Art. 98.

For the pipe BC we have

$$H_1-H_2=\frac{v^2}{2g}+10-\frac{v'^2}{2g};$$

$$H'=f\frac{1200}{d'}\frac{v'^2}{2g};$$

hence
$$\frac{v^2}{2g}+10-\frac{v'^2}{2g}=f\frac{1200}{d'}\frac{v'^2}{2g}.$$

The value of d having been previously determined, v is known. And since $v'=\frac{8}{\pi d'^2/4}$, d' may be determined by solving the last equation in the usual way. The branch BD may be treated in the same manner.

EXAMPLE.

Complete the solution of the above numerical case.

101. Relation of Pipe to Hydraulic Gradient.—For a uniform pipe, straight and unobstructed, the hydraulic gradient (Art. 86) is uniform for the entire length. If the stream discharges into the air without contraction, the pressure at the outlet end is atmospheric, while near the intake end the gradient falls from the level of the reservoir surface an amount equal to the velocity head plus the entrance loss. Between these points the hydraulic gradient is a straight line, as shown in Fig. 57. Gradual curves in the pipe line will not materially affect the uniformity of the hydraulic slope, but the velocity which corresponds to a given total fall will be less when there are bends than in a straight pipe of the same length.

In laying a long pipe line, both horizontal and vertical curves must often be introduced because of the contour of the ground. In planning the line it is of especial importance to see that

the pipe shall everywhere lie below the hydraulic gradient as it will exist when the rate of discharge has its greatest value.

Fig. 60 shows a case in which a large part of the pipe is above the hydraulic gradient *AB*. Assuming full flow to exist, the pressure in the pipe everywhere between *C* and *B* is less than atmospheric. The resultant pressure upon the pipe is thus from without, but (aside from the danger of collapsing the pipe) the decrease of pressure will not of itself interfere with the flow unless the pipe goes high enough to reduce the pressure to absolute zero. Practically, however, it would be difficult to start full flow under such conditions, and even if started it would not continue unless the pipe were absolutely air-tight

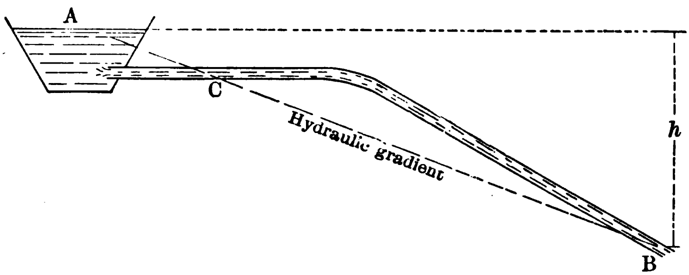


FIG. 60.

and the water free from air. Since air is always carried by natural waters, there would soon be an accumulation of air in the higher parts of the pipe which would check the flow even if there were no leakage from without.

102. Effect of Air in Checking Flow.—The collection of air at a summit may seriously interfere with the discharge of a pipe even when everywhere below hydraulic gradient, but so long as the pressure is above that of the atmosphere the air may be removed by means of a valve provided for the purpose.

To understand the action of air in checking the flow, consider the case represented in Fig. 61. Here the hydraulic gradient for full flow is everywhere above the pipe. But if air collects in the bend *C*, the stream will finally be divided into two parts, the surface beyond *C* being forced down toward *D*,

as shown at Y. The flow will be retarded, but will not be stopped unless the pressure of the confined air becomes great enough to balance the static pressure due to the water column AX plus atmospheric pressure. If D is far enough below B, the condition represented in Fig. 62 will finally ensue and flow will cease. The pressure of the confined air will then just balance each of the two equal * columns AX, YB.

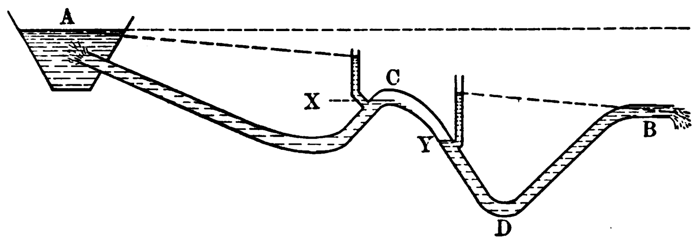


FIG. 61.

The effect of air upon the hydraulic gradient is shown in Figs. 61 and 62. With full flow the gradient would be a line of nearly uniform slope from A to B. As air accumulates, the pressure increases in the up-stream part of the pipe and decreases in the down-stream part, piezometer columns standing

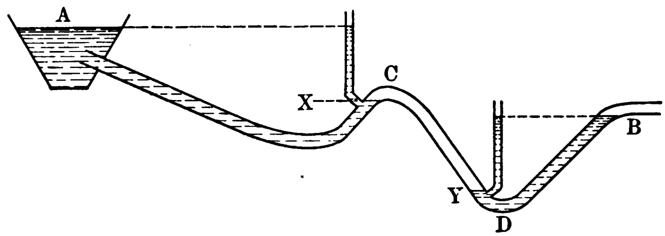


FIG. 62.

at equal heights above the surfaces X and Y as shown in Fig. 61. When the flow is stopped, as in Fig. 62, the gradient consists of two horizontal lines, one lying in the plane of the reservoir surface, the other in a horizontal plane as far above Y as X is below the reservoir surface.

* This neglects the weight of the confined air, by reason of which the pressure at Y would be slightly greater than that at X.

103. Long Pipe with Full Discharge.—In case of a long pipe the friction loss, expressed by the term $f\frac{l}{d}\frac{v^2}{2g}$, is very great in comparison with the other losses of head, and also in comparison with the velocity head. Equation (1) therefore reduces practically to the form

$$h=f\frac{l}{d}\frac{v^2}{2g},$$

and equation (5) to the form

$$h-y=f\frac{l}{d}\frac{v^2}{2g}.$$

In the majority of practical problems relating to the design of water supply systems these simplified equations are used, the value of the neglected terms being within the limits of reliability of the data.

104. Chézy's Formula.—The formula for loss of head in a uniform straight pipe, given in Arts. 79 and 80, is often written in another form. Introducing the hydraulic radius (the ratio of the area of the cross-section to its circumference) instead of the diameter, the formula

$$H'=f\frac{l}{d}\frac{v^2}{2g}$$

becomes

$$H'=f\frac{l}{4r}\frac{v^2}{2g}.$$

Solving for v, writing s for $\frac{H'}{l}$, and introducing a new coefficient c such that

$$c=\sqrt{\frac{8g}{f}},$$

we have

$$v=c\sqrt{rs},$$

which is known as Chézy's formula.*

* This is the formula commonly used in the discussion of flow in open channels. See Chapter XI.

If there are bends in the pipe, or if the roughness of the surface varies in different parts, the losses of head in different equal lengths are unequal, and the hydraulic slope s varies along the pipe. This is usually the case in practice. In such cases the formula $v=c\sqrt{rs}$ is still often employed, s being regarded as the average hydraulic slope for the entire length, i.e., $s=H'/l$, where H' is the total loss of head in the length l. The value of c must be taken less than in the case of straight pipe.

The formula for f given in Art. 80, based upon Darcy's results, is equivalent to the following formula for c:

$$c=113.6\sqrt{\frac{d}{d+.0836}}.$$

Further discussion of the values of f and c is given in Chapter IX.

The two formulas

$$H'=f\,\frac{l}{d}\,\frac{v^2}{2g},$$

$$v=c\sqrt{rs},$$

are, as above shown, identical. Although the latter has been much employed, the former is preferable for the reason that f is an abstract number, while c depends upon the system of units employed.

Since for a circular cross-section of diameter d

$$F=\frac{\pi d^2}{4} \quad \text{and} \quad r=\frac{d}{4},$$

Chézy's formula may be written

$$v=\frac{c}{2}\sqrt{sd},$$

and we have also

$$q=Fv=.393cs^{\frac{1}{2}}d^{\frac{5}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

or

$$d^{\frac{5}{2}}=2.54\frac{q}{c\sqrt{s}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

EXAMPLES.

1. A pipe 18″ in diameter will discharge what quantity of water per second if the hydraulic gradient falls 10′ in a length of 8000′?

Ans. $q = 4.22$ cu. ft. per sec.

2. A pipe 8500′ long is to discharge 40 cu. ft. per sec., with a total fall of 25′ in the hydraulic gradient. What should be the diameter?

Ans. About 3.1′.

3. If the pipe in Ex. 2 is designed so as to just satisfy the conditions stated, how much will the hydraulic gradient fall if the rate of discharge becomes 50 cu. ft. per sec.? *Ans.* About 39′.

4. If two equal pipes are to be substituted for the one pipe in Ex. 2, what diameter should they have? *Ans.* About 28″.

CHAPTER IX.

FRICTIONAL LOSS OF HEAD IN PIPES.

105. Importance of Frictional Loss.—The so-called "friction head," or frictional loss of head in pipes, is the most important factor affecting the discharging capacity of pipe lines or distribution systems in practice, since other losses are commonly small in comparison with it. The present chapter will therefore be devoted to a fuller discussion of methods of determining or estimating this loss.

106. Method of Measuring Loss of Head.—The measurements which must be made in order to determine the actual loss of head in a given length of pipe in which a condition of steady flow exists have been indicated in Art. 73. The effective head at any section involves the three quantities elevation, pressure, and velocity of flow. Of these the last is often the most difficult to measure with the requisite accuracy. Methods of accomplishing it are considered briefly in Chapter XIII. If the cross-sectional areas of the pipe at the two sections at which the effective head is to be measured are equal, the velocity terms disappear from the value of the lost head; but even in this case the velocity corresponding to any measured loss of head must be known if the results are to have value as a guide to design.

The measurement of the difference in elevation of the two cross-sections, with the accuracy requisite in experimental work of this kind, may involve considerable labor, especially in the case of long pipe lines laid under the conditions of actual practice. The necessity of making such measurements may be

obviated if it is possible to shut off the flow and read the pressure gauges when hydrostatic conditions exist throughout the pipe. Thus, let AB (Fig. 63) be the length of pipe in which

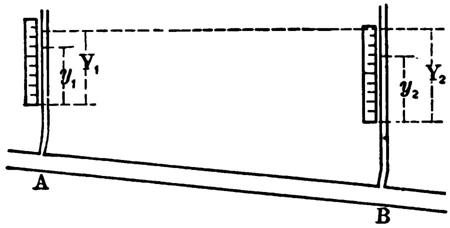


FIG. 63.

loss of head is to be determined, and suppose piezometer tubes to be connected at A and B. Let the position of the top of each piezometer column be read upon a fixed vertical scale, y_1 and y_2 being simultaneous values of the two readings. Let Y_1, Y_2 be values of y_1, y_2 when the flow is cut off.

When flow takes place the loss of head in the entire line above A is

$$Y_1 - y_1 - \frac{v_1^2}{2g},$$

while the loss in the entire line above B is

$$Y_2 - y_2 - \frac{v_2^2}{2g}.$$

Hence the loss from A to B is

$$H' = \left(Y_2 - y_2 - \frac{v_2^2}{2g}\right) - \left(Y_1 - y_1 - \frac{v_1^2}{2g}\right).$$

If $v_1 = v_2$,

$$H' = (y_1 - y_2) - (Y_1 - Y_2).$$

Thus the actual heights of the piezometer columns above a common datum need not be measured, the above value of H' being independent of the positions of the zero points of the two fixed scales.*

* If the zero points are so located that $Y_1 < Y_2$, a constant may be added to all values of y_1 such that $Y_1 - Y_2$ will be positive; every value of $y_1 - y_2$ will then be positive.

The same method is applicable if any other kind of pressure gauge be used instead of a simple water piezometer, and the above formula for H' still holds if y_1 and y_2 denote simultaneous readings of the two gauges, and Y_1, Y_2 the values of y_1, y_2 under static conditions; it being understood that all gauge readings are in feet of water. If the gauges are otherwise graduated, either the readings or the final value of H' may be reduced to equivalent water column by applying the proper factor.*

107. Formulas for Friction Loss.—It appears not to be possible to express the frictional loss of head in pipes accurately, in any simple way, in terms of the several quantities upon which its value depends. For straight pipes of a given kind the loss in a given length varies chiefly with diameter and velocity of flow. Many attempts have been made to establish a formula expressing its value in terms of these two variables, which should be applicable to pipes of a given kind throughout the entire range of diameters and velocities likely to be met in practice, and which could be applied to different kinds of pipe by a proper choice of constants. Hitherto such attempts have not been successful, and there is little reason to expect future success in this direction. The most careful experiments made with pipes under practical conditions show that the so-called friction loss is materially influenced by factors which cannot be controlled by the experimenter, and which in the practical use of the pipes are sure to vary. One of these factors is temperature. Another is the character of the pipe surface, which may vary from month to month, or even from day to day,† enough to appreciably change the loss of head due to a given

* If a mercury manometer of the kind described in Art. 75 is employed, the fluctuation of the surface of the mercury in the reservoir must be taken into account in reducing gauge-reading to equivalent water column. Also, in case the pressure is so great that the column of mercury is of considerable length, it may be necessary to apply a correction for changes of temperature, for which purpose the length of the column must be known approximately. See Trans. Am. Soc. C. E., Vol. XL, p. 481; Vol. XLIV, p. 39.

† Darcy, Recherches expérimentales, p. 107.

velocity. In view of these facts it seems futile to seek a single general formula by which the friction loss in any proposed pipe line can be predicted with great accuracy.

Most of the formulas that have been proposed have a certain basis of theory. This theory, and some of the more important of the formulas, will be considered in the following articles.

108. Theoretical Basis of Formulas for Friction Loss.— Consider a straight pipe whose cross-section is uniform but of any shape. Let F denote the area and C the perimeter of the cross-section, and let $r=F/C$ (as in Art. 79). Let A and B (Fig. 64) be two cross-sections whose distance apart is l, and

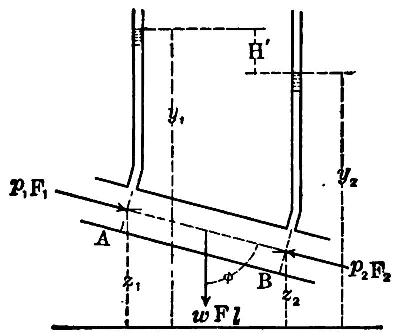


FIG. 64.

let p_1, p_2 denote the pressures at centroids of sections A and B respectively. Conceive piezometer tubes communicating with the pipe at the two sections, and let y_1, y_2 denote elevations of the two water columns above a chosen datum plane; z_1, z_2 being heights of centroids of the two sections above the same datum.

The body of water between A and B being in a condition of steady flow, the external forces acting upon this body are in equilibrium. Let these be resolved in the direction of the flow.

The system of external forces consists of the normal pressures exerted by the adjacent water upon the two cross-sectional areas A and B, the weight of the body of water AB, and the normal and tangential forces exerted by the pipe surface, the tangential or frictional forces being opposite in direction to the

flow. Let P denote the sum of the frictional forces on the entire area of contact of the body AB with the pipe.

Since in any cross-section of the stream the pressure varies according to the hydrostatic law, the total pressure on the cross-section is equal to the product of the area into the intensity of pressure at the centroid (Art. 17). Hence

$$p_1F = \text{total pressure on cross-section } A;$$
$$p_2F = \text{ " \quad " \quad " \quad " } \quad B.$$

The weight of the body of water between A and B is wFl, and its component in the direction of the flow is

$$wFl \cos \phi = wF(z_1 - z_2).$$

Equating to 0 the sum of the resolved forces in the direction of the flow,

$$p_1F - p_2F + wF(z_1 - z_2) - P = 0.$$

Since $p_1/w + z_1 = y_1$ and $p_2/w + z_2 = y_2$, and since $y_1 - y_2 = H' =$ loss of head between A and B, the equation may be written

$$H' = y_1 - y_2 = \frac{P}{wF}.$$

Thus the loss of head is expressed in terms of the total frictional force exerted by the pipe surface upon the water.

If the frictional force per unit area is P', so that

$$P = ClP',$$

we may write

$$H' = \frac{ClP'}{wF} = \frac{l}{r} \cdot \frac{P'}{w},$$

r being the hydraulic radius as already defined (Art. 79).

Experiment indicates that the value of P' is independent of the pressure, but varies with the velocity of the relative motion. If the law of variation were known, the above formula would

give the loss of head in terms of the surface velocity. Thus, if v_1 is the surface velocity, and if

$$\frac{P'}{w}=\phi(v_1),$$

the formula becomes

$$H'=\frac{l}{r}\phi(v_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

109. Introduction of Mean Velocity in Place of Surface Velocity.—Equation (1) would be of little use even if the form of the function $\phi(v_1)$ were known, since the quantity of importance is not the surface velocity v_1, but the mean velocity v.

If it be assumed that there is a fixed relation between v_1 and v which is independent of the size of the pipe, there results an equation of the form

$$H'=\frac{l}{r}F(v). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

If this assumption were correct, the dependence of H' upon the size of the pipe would be expressed by a very simple relation. Thus, since for circular pipes $r=d/4$, loss of head would vary inversely as diameter. The fact that this simple law is not verified by experiment indicates that the relation between mean velocity and surface velocity is not independent of the size of the pipe. Some of the earlier writers on Hydraulics, however, assumed that mean velocity varies directly as surface velocity, and thus deduced the form of the function $F(v)$ in equation (2) from the experimental law governing the friction between a solid and a fluid.

110.—Application of Experimental Law of Fluid Friction.—The frictional force per unit area exerted by a solid body upon a fluid flowing uniformly over its surface is found (*a*) to be independent of the pressure and (*b*) to vary approximately as the square of the velocity of flow, except for quite small velocities,

and (c) to vary nearly in direct ratio with the velocity of flow if this is very small.

Applying this to the pipe problem, the value of P' may be expressed approximately in the following form:

$$P' = a'v_1 + b'v_1^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Assuming that v_1 varies directly as v, equation (2) may be written

$$H' = \frac{l}{r}(av + bv^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

111. Formula of Prony.—Equation (4) is the formula given by Prony, who, from such experimental data as were available, deduced numerical values of a and b which were supposed to hold for all sizes of pipe. It was supposed, also, that the character of the surface of the pipe had little influence upon the values of these coefficients.

112. Formula of Darcy.—Equation (4) was also employed by Darcy, who found, however, that the coefficients a and b vary greatly with the character of the pipe surface,* and also that they are not independent of the diameter. From his experiments he deduced empirical formulas of the forms

$$\left.\begin{aligned} a &= \alpha + \frac{\beta}{d^2}, \\ b &= \alpha' + \frac{\beta'}{d}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

giving numerical values for α, β, α', β', applicable to pipes of about the degree of smoothness of new cast iron.

Darcy's experiments, as well as many made subsequently, indicate not only that the term av in equation (4) is unimportant in comparison with bv^2 except for quite small velocities, but also that its importance is less the rougher the surface of

* Being, for example, about twice as great for cast-iron pipes fouled by some years of use as for the same pipes new or thoroughly cleaned.

the pipe. For pipes in practical use, and for the range of velocities ordinarily existing, Darcy recommended as sufficiently correct a formula of the form

$$H'=\frac{l}{r}b_1v^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

And for values of d within the range of his experiments (from 0.5 inch to 20 inches approximately) he gave the following formula for the coefficient b_1:

$$b_1=\alpha+\frac{\beta}{d}.$$

The equivalent of this formula, with numerical values of the coefficients, is given in Art. 80.

113. Chézy's Formula.—It will be seen that equation (6) is really the same as the formula already given in two forms (Arts. 80, 104):

$$H'=f\frac{l}{d}\frac{v^2}{2g}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

$$v=c\sqrt{rs}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

In the latter form it is often known as Chézy's formula, and is identical with the formula commonly employed in estimating the discharge of streams flowing in open channels.*

114. Kutter's Formula.—Kutter's formula is an empirical expression for the value of c in terms of r, s, and a certain quantity n called the coefficient of roughness, whose value depends upon the character of the surface over which the water is flowing. The relation is as follows:

$$c=\frac{\frac{1.811}{n}+41.65+\frac{.00281}{s}}{1+\frac{n}{\sqrt{r}}\left(41.65+\frac{.00281}{s}\right)}, \quad . \quad . \quad . \quad . \qquad (10)$$

r being in feet and v in feet per second.

* See Art. 128.

Although this complex formula was based upon experimental data for flow in open channels, it has often been applied to pipes and closed conduits, especially those of large size. Its use cannot, however, be recommended, except in the absence of experimental knowledge directly applicable to the case in hand.

Some indication of the values of n to be used for pipes of different kinds will be given below.

115. Exponential Formula.—The foregoing discussion has brought out the fact that the loss of head in a given length of pipe varies approximately, but not exactly, as the square of the velocity of flow in pipes of the same size, and inversely as the diameter in pipes of different sizes; all being assumed alike in the character of the interior surface. It was found by Osborne Reynolds,* in experiments with lead tubes of one-fourth inch and one-half inch diameter, that the loss of head varied as a power of the velocity whose index was less than 2. A similar relation was observed by Lampe in experiments with a pipe about 16.5 inches in diameter. From a study of the experimental results of Darcy and Poiseuille, Reynolds found a similar exponential relation, but with the index varying with the character of the pipe. A study of other experimental data obtained with both small and large pipes shows that the variation of loss of head with velocity in a given pipe may often be closely represented by the formula

$$H' = Av^m,$$

with constant values of A and m. The value of m is usually less than 2, but varies considerably for different pipes, and in some cases has been found to be greater than 2. There is some evidence tending to show that for a given character of pipe surface m is independent of the diameter, and that it increases with the roughness of the surface. Thus a value of 1.72 was found by Reynolds for the small lead tubes, and about the same value has been found for other quite smooth

* Philosophical Transactions of the Royal Society, 1883, Part III, p. 975.

pipes,* while for pipes quite rough by incrustation or otherwise a value of about 2 has been found.†

Whether the character of the pipe surface is the only important factor affecting the value of the exponent cannot be determined from known experimental data, but it seems likely that for pipes laid under the conditions of practice it may vary with the curvature of the pipe line, the temperature of the water, and perhaps other factors.

The value of A probably depends upon the character of the pipe surface, as well as upon the diameter. A series of experiments upon pipes alike in all respects except in size would doubtless show a regular variation of A with the diameter. It has been supposed that this also may prove to be an exponential relation, but for large pipes experimental data for the establishment of such a law are almost wholly lacking. Such evidence as there is seems to show that if A varies with a power of d, the exponent will be negative and numerically greater than unity.‡

It was pointed out by Reynolds that the above formula is a convenient one for representing experimental data because of the ease with which the constants can be determined graphically. Taking logarithms, the formula becomes

$$\log H' = \log A + m \log v.$$

If a series of experimental values of H' and v satisfies this formula, the locus whose coordinates are $\log H'$ and $\log v$ will be a straight line. The slope of this line determines the value of m, and its intercept on the axis along which $\log H'$ is measured gives the value of $\log A$. The plotting is facilitated by the use of logarithmic cross-section paper. The plotting of the results in this manner shows at once whether the assumed form of formula agrees well with the experimental data, and also gives a simple determination of the constants in case it does.

* See, for example, Trans. Am. Soc. C. E., Vol. LI, p. 253.

† Phil. Trans. Roy. Soc., 1883, Part III, p. 981. Trans. Am. Soc. C. E., Vol. XXXV, p. 258.

‡ Tables for practical use, based upon an exponential formula, have recently been published by Gardner S. Williams and Allen Hazen.

116. Critical Velocity.—The experiments of Reynolds showed that there was a certain velocity below which the loss of head varied nearly as the first power of the velocity, and above which it varied as a higher power. The value at which the law changes he called the critical velocity. By observations of the flow in glass tubes he found that for very low velocities the flow took place quite accurately in parallel filaments, while above the critical velocity the stream became turbid.

In practical Hydraulics velocities below the critical value rarely or never need be considered. It is to the range of velocities above the critical point that the above discussion of the exponential formula refers.

117. Formula Adopted.—For the purpose of expressing working rules for estimating loss of head, the equivalent formulas

$$H' = f\frac{l}{d}\frac{v^2}{2g},$$

$$v = c\sqrt{rs},$$

will here be employed. For convenience values both of the friction factor f and of the coefficient c will in most cases be given. When one of these quantities is known the other can be computed from the relation

$$f = \frac{8g}{c^2}.$$

As already pointed out, f is an abstract number and therefore independent of the system of units employed, while c depends upon the units of length and time, c^2 being of the same dimensions as g.

Owing to the imperfection of the theory upon which the above formulas are based, f and c cannot be regarded as constants for a given kind of pipe, but vary with both diameter and velocity.

118. Values of Friction Coefficients.—Many attempts have been made, by study of known experimental data, to establish

definite values of the friction factors for the kinds of pipe used in practical hydraulic works. Formulas or tables professing to give such values with great accuracy cannot, however, be accepted with confidence, because of the great uncertainties in the reliability of the data upon which they are necessarily based.

The experiments of Darcy * are probably the most trustworthy series yet made covering any considerable range of diameters and velocities. Upon the results he based two formulas, one expressing as accurately as possible the variation of the friction factor with both diameter and velocity, the other neglecting the variation with velocity. The latter, which was recommended by its author for practical use, was equivalent to the following, in which d is the diameter in feet:

$$f = .0199 + \frac{.00166}{d}.$$

This is for pipes of new or clean cast iron, or other pipes of equal smoothness. For pipes tuberculated or otherwise roughened by some years of use Darcy recommended that the values of f given by this formula be doubled.

The pipes used in Darcy's experiments ranged in diameter from 0.0122 meter to 0.5 meter. The most reliable experiments made since by others indicate that his formula gives safe values, not only within this range, but for larger sizes, and the formula has often been used for diameters as great as 4 feet. Table III (page 114), based upon the formula, is carried to about double the largest size experimented upon by Darcy.

There is some evidence that the coefficients in column 3 of the table are somewhat too great for pipes as smooth as ordinary clean cast-iron water pipes. It is, however, important in engineering design to allow a margin of safety to cover both the large element of uncertainty in existing knowledge regarding friction losses in pipes of any given character, and the uncertainty in any given case regarding the actual present

* Recherches expérimentales relatives au mouvement de l'eau dans les tuyaux. Henry Darcy. Paris, 1857.

and future character of the particular pipe under consideration. The tabulated coefficients, both for smooth and for rough pipes, probably err on the side of safety, unless it be for the case of pipes which have suffered exceptional deterioration with long use.

TABLE III.

VALUES OF THE FRICTION FACTOR f AND OF THE COEFFICIENT c, BASED UPON THE FORMULA OF DARCY.

1	2	3	4	5	6
Diameter.		Smooth Pipe		Rough Pipe	
Inches.	Feet.	f	c	f	c
1	.0833	.0398	80	.0796	57
2	.167	.0298	93	.0596	66
3	.250	.0265	99	.0530	70
4	.333	.0248	102	.0496	72
6	.500	.0232	105	.0464	74
8	.667	.0224	107	.0448	76
10	.833	.0219	108	.0438	77
12	1.000	.0216	109	.0432	77
14	1.167	.0213	110	.0426	78
16	1.333	.0211	110	.0422	78
18	1.500	.0210	111	.0420	78
24	2.000	.0207	111	.0417	79
30	2.500	.0206	112	.0412	79
36	3.000	.0205	112	.0410	79

The degree of roughness or smoothness of any given pipe must be a matter for the judgment of the engineer. But, except in the rare cases in which pipes are laid for temporary use only, the design must be governed by probable future conditions rather than by the character of the pipe when new.

119. Large Pipes.—Although there is little experimental evidence regarding the variation of f and c with the diameter in the case of large pipes, this variation appears to be unimportant. As regards variation with the velocity, there is evidence that, except for quite small velocities, this is of little practical importance even with smooth pipes, and is quite inappreciable in the case of rough pipes, such as those of riveted steel or almost any pipe after long use. For pipes from 3 ft.

to 6 ft. in diameter it is therefore sufficient to use values of f and c which are independent of velocity and diameter, varying only with the character of the pipe.

The general range of these values is indicated in the following table. It is of course to be understood that actual pipes present all gradations of roughness, and the four cases specified in the table can only serve as a very general and approximate guide. The values tabulated (except for Case I) are designed to allow for the resistance due to such curves as are likely to exist in practical cases.

TABLE IV.

FRICTION FACTORS AND COEFFICIENTS FOR LARGE PIPES.

(Diameter from 3 ft. to 6 ft.)

Case.	Typical Pipe.	f	c	n
I. (Very smooth.)	Exceptionally smooth cast-iron pipe, new or thoroughly cleaned, without bends.	.013	140	.011
II. (Smooth.)	Ordinary new cast-iron or wooden pipe, with some bends.	.018	120	.0125
III. (Rough.)	New riveted pipe, with some bends, or any pipe after some years of use.	.023	106	.014
IV. (Very rough.)	Any pipe after long use.	.046	75	.019

120. Kutter's Coefficient of Roughness.—Kutter's formula (Art. 114) has come into quite general use as a practical guide in estimating the discharging capacity of large pipes. This formula assumes to take account of the roughness of the surface by the value assigned to the coefficient n. It also makes c vary with diameter and velocity. The values of n given in the last column of the above table are about the average values implied by the corresponding values given for c.

CHAPTER X.

EQUATION OF ENERGY FOR STREAM OF LARGE CROSS-SECTION.

121. Streams of Large Cross-section.—The theoretical discussion given in Chapter IV, leading to the general equation of energy for a steady stream, involved the assumption that the intensity of pressure may be regarded as uniform throughout any cross-section, and that all particles passing the section have equal velocities. These assumptions cannot be supposed to be even approximately true when the cross-section is large, as in the case of rivers and canals. Let us consider whether the theory requires important modification when variations of pressure and of velocity in a cross-section are taken into account.

122. Variation of Pressure Throughout a Cross-section.—If all particles move in straight lines parallel to the axis of the stream, the pressure in any normal cross-section varies with the depth according to the hydrostatic law. For let A and B be two points such that AB is perpendicular to the direction of flow, and consider a prism of water of small cross-section whose axis is AB. Resolving all forces acting upon this prism in the direction AB, the sum of such resolved forces must be zero, since there is no acceleration in this direction. Hence the reasoning of Art. 11 may be applied, leading to the same conclusion.

123. Variation of Velocity in a Cross-section.—The way in which the velocity varies throughout the cross-section has been the subject of considerable investigation, both experimental and theoretical. The following discussion is, however, independent of any assumption as to the actual distribution of

velocities; though, as will be seen, a certain term in the equation of energy cannot be evaluated unless the law of distribution is known.

124. Energy Passing a Cross-section.—To determine the energy passing any cross-section per unit time, we may reason substantially as in Art. 59, introducing such changes as are

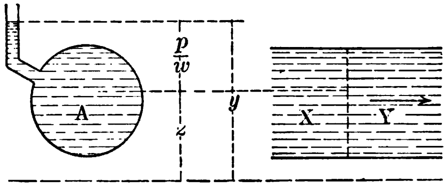


FIG. 65.

necessitated by the variation of pressure and of velocity throughout the section. Referring to the cross-section A (Fig. 65), let

v = velocity at any point in the section;
F = whole area of the section;
q = rate of discharge across area F;
$v' = \frac{q}{F}$ = mean velocity;
p = pressure at point whose velocity is v;
z = height of that point above datum.

Energy transferred across the section by pressure.—The energy transferred across an element dF of the cross-section by reason of the pressure and velocity may be computed as in Art. 59. Its value per second is $vp\,dF$, and therefore the energy thus transferred across the entire section per second is

$$\int pv\,dF, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

the integration covering the whole section F.

Potential energy carried across the section.—The potential energy carried across an elementary area dF per second is

$$wzv\,dF,$$

and the potential energy passing the whole area F per second is

$$w\int zv\,dF. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

Sum of potential energy and energy transferred by pressure.—The sum of the values (1) and (2) may be written

$$w\int\left(\frac{p}{w}+z\right)v\,dF.$$

Now since the pressure varies throughout the section according to the hydrostatic law, in passing to different points in the section p/w increases (or decreases) by just the amount of decrease (or increase) of z; that is, $z+p/w$ is constant throughout the section. Denoting this sum by y, it is seen that y is equal to the height above datum of the top of a piezometer column communicating with the pipe at some point of the given cross-section.

Since y is constant in the integration, the value of the integral becomes

$$wy\int v\,dF=wyq=wyv'F. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

Kinetic energy passing the section.—The amount of kinetic energy passing the elementary area dF in one second is

$$\frac{wv^3}{2g}dF,$$

hence the amount passing the whole section F is

$$\frac{w}{2g}\int v^3dF. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

This integral cannot be evaluated unless the variation of v throughout the section is known. It may, however, be shown that it is greater than the value which would be obtained if

the mean velocity v' were assumed to apply throughout the section. That is,

$$\frac{w}{2g}\int v^3 dF > \frac{wFv'^3}{2g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Thus, let $v = v' + u$; then

$$\int v^3 dF = \int (v'^3 + 3v'^2 u + 3v' u^2 + u^3) dF$$
$$= v'^3 F + 3v'^2 \int u\, dF + \int (3v' u^2 + u^3) dF.$$

But $$\int u\, dF = 0,$$

since u denotes the excess of actual velocity at any point over mean velocity. We may therefore write

$$\frac{w}{2g}\int v^3 dF = \frac{wFv'^3}{2g} + \frac{w}{2g}\int (3v' u^2 + u^3) dF. \quad . \quad . \quad (6)$$

The last integral is positive; for

$$3v' u^2 + u^3 = 3u^2 \left(v' + \frac{u}{3}\right);$$

and since $v' + u$ or v is always positive even when u is negative, it follows that $v' + \frac{u}{3}$ must be positive for all values of u. The inequality (5) is thus proved.

The total kinetic energy carried across the section per unit time may therefore be written

$$\frac{wqv'^2}{2g} + K, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

K being a positive quantity equal to the last term of (6).

Total energy passing the section.—The total energy passing

the section in one second is the sum of the values (3) and (7). It may be written

$$wq\left(y+\frac{v'^2}{2g}+k\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

in which $k=K/wq$.

The value of the energy passing any cross-section *per unit weight of water discharged* may therefore be expressed in either of the following forms:

$$y+\frac{v^2}{2g}+k=z+\frac{p}{w}+\frac{v^2}{2g}+k, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

in which v is now written for the mean velocity in the cross-section. As shown above, z and p may refer to any point in the section, their sum being constant.

Comparing this result with that reached in Art. 59 it is seen that the effect of the variation of velocity is represented by the term k.

125. Equation of Energy in Case of Large Stream.—The reasoning of Art. 60 may be applied to the case of a large stream, using the expression above deduced for energy passing a section. With suffixes ($_1$) and ($_2$) to refer to up-stream and down-stream sections respectively, we have

$$z_1+\frac{p_1}{w}+\frac{v_1^2}{2g}+k_1=z_2+\frac{p_2}{w}+\frac{v_2^2}{2g}+k_2+H',$$

in which the values of z and p refer to any point of the section. Instead of $z+p/w$ we may write y; y being constant for each section, and meaning the height above datum of the top of a piezometer column communicating with the pipe at any point of the perimeter of the section.

As shown above, k_1 and k_2 are positive quantities, the value of each depending upon the distribution of velocities in the corresponding section. The quantity k cannot be zero unless the velocity has the same value throughout the section, and will be great in proportion as the variation of velocity is great

If the two sections compared have the same size and shape, it may reasonably be assumed that $k_1 = k_2$; in this case the equation of energy takes the same form as if the velocity were uniform throughout every cross-section. For sections of unequal size and consequent unequal mean velocities no such assumption can rationally be made.

Ordinarily, for circular pipes, no great error results from using the equation of energy in the ordinary form, as above in the discussion of small pipes. This statement is justified by experiment; and it may be concluded either that the value of k is usually small, or that its values for sections of different size are nearly equal.

CHAPTER XI.

UNIFORM FLOW IN OPEN CHANNELS.

126. General Principles.—The principles employed in the discussion of flow in pipes apply in the main to streams in open channels. The chief difference arises from the fact that in the case of a stream in an open channel the upper surface is in contact with the atmosphere. From this it follows (*a*) that the pressure at this surface has a known constant value, and (*b*) that the frictional resistance to flow at this surface is much less than at the surface of the channel itself.

The way in which the velocity of flow varies throughout any cross-section is in general unknown, and can be determined only by experiment. The effect of this variation upon the general equation of energy for steady flow has been considered in Chapter X, the discussion there given applying to open as well as to confined streams. In the following discussion it is usually the mean velocity in a cross-section, rather than the actual velocities in different parts of the section, that will be considered. If this mean velocity is denoted by v, the relation

$$\begin{aligned} q &= Fv = F_1v_1 = F_2v_2 \\ &= \text{constant for all cross-sections} \end{aligned}$$

holds for an open channel, if the flow is steady.

The following discussion will be restricted to the case of steady flow in a channel of uniform cross-section and slope. In the cases of most importance it will further be true that the depth of the water is the same at all sections, so that the water cross-sections are all alike in size and shape. The flow

is then said to be *uniform*. The problem of *non-uniform flow* will be considered in the next chapter.

127. Uniform Flow.—Consider the case of a channel of uniform cross-section and slope and of considerable length, into the upper end of which water is admitted at a constant rate. In a short time the flow will become *steady*, and it will also become practically *uniform* throughout the greater part of the straight and uniform channel under consideration. When this condition is reached, the velocity of flow will depend upon (*a*) the slope of the channel, (*b*) the character of the surface over which the water flows, and (*c*) the dimensions of the cross-section.

(*a*) The velocity is greater as the slope is greater.

(*b*) The velocity is greater as the channel surface is smoother.

(*c*) The greater the length of the wetted perimeter of the cross-section in comparison with its area, the greater the retarding effect of friction, and therefore the less the velocity.

128. Formula for Mean Velocity in Case of Uniform Flow.—No theoretical discussion of flow in open channels, even in the simplest case of uniform flow, can serve as more than a rough guide in the solution of practical problems. The formula most commonly employed in practice is that of Chézy, already given as applying to pipes. The reasoning of Arts. 108–110 may be applied with slight change to the case of an open channel.

Let Fig. 66 represent a longitudinal section of the stream, and consider the body of water between two cross-sections at

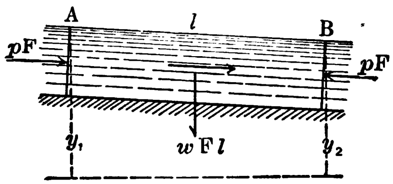


FIG. 66.

A and *B*. The flow being steady, this body has no acceleration, and the forces acting upon it form a system in equilibrium.

Let these forces be resolved in the direction of the flow. The forces are the following:

(*a*) The weight of the body, equal to wFl, F being the cross-sectional area and l the length AB.

(*b*) The pressures of the adjacent water upon the cross-sections at A and B. Since the two sections are alike in all respects, and since the upper surface is under uniform pressure, the total pressure upon the cross-section A is equal and opposite to that upon B, each having the value pF, the product of the cross-sectional area into the intensity of pressure at the centroid of the area.

(*c*) The frictional forces exerted in the direction BA upon the water flowing over the channel surface. Computing it as if all particles of the water had equal velocities, and assuming the friction per unit area P' to be independent of the pressure and proportional to the square of the velocity, the total frictional force would be

$$P = ClP' = Cl \cdot kv^2,$$

C being the length of the wetted perimeter, and k a constant depending upon the roughness of the surface.

(*d*) The frictional retardation due to air at the upper surface. This is assumed to be negligible.

Equating to zero the sum of the resolved parts of these forces in the direction AB, and representing by s the sine of the angle between the water surface and the horizontal, we have

$$wFls - kClv^2 = 0.$$

Introducing $r = F/C$ = hydraulic radius of the cross-section, the equation may be written

$$v = c\sqrt{rs},$$

in which c is a coefficient whose value depends upon the roughness of the channel, and would be constant for any given kind of channel if the assumptions above made were rigorously true.

In comparing this result with that previously obtained for pipes (Art. 113), it will be noticed that s here means the *slope*

of the water surface, while in the previous case it means the *hydraulic slope.* It is obvious, however, that the hydraulic slope of a stream having a free upper surface is identical with the slope of that surface. If a piezometer be introduced at any cross-section, as in Fig. 67, the water will rise in it to the level of the surface of the stream. Fig. 66 shows at once that

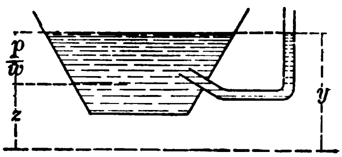


FIG. 67.

$$s=\frac{y_1-y_2}{l}=\frac{H'}{l},$$

so that the above equation might be written in either of the two forms given in Art. 113.

~~129.~~ Variation of Coefficient in Chézy Formula.—The above theory is obviously seriously defective, but no general formula having a better basis in theory has been proposed. In accepting this formula as a practical guide, the coefficient c must be regarded as dependent not merely upon the character of the channel surface, but upon various other conditions. The practical rules which have been given for estimating the value of c, based upon experiment, usually express it as a function of one or more of the three quantities v, r, and s, in addition to the roughness of the channel. It is altogether improbable that any such assumption can properly be made (except as a rough approximation) in comparing streams having very different forms of cross-section, since the form of the section probably affects the flow in a manner which is not dependent merely upon the value of r, and which in fact cannot be expressed accurately in any simple way. In solving practical problems in flow, however, a rough approximation to the true result is often all that it is possible to attain.

In spite of the defects in the theory above given, it is still true that the most important factor affecting the value of c is the character of the channel as regards roughness. The range of values of the coefficient for different cases may be stated roughly as follows:

$c=100$ to 140 for timber or cement-lined conduits.
$c=80$ " 110 " conduits of smooth masonry.
$c=60$ " 90 " conduits of rough masonry (rubble).
$c=50$ " 80 " ditches in clean earth or gravel.
$c=20$ " 40 " ditches or canals in bad order.

The experimental determination of c is much more difficult for open channels than for pipes. An accurate determination requires that the channel under experiment shall be quite truly uniform in slope; otherwise the cross-section of the stream will vary even if that of the channel does not. Unless the condition of uniform flow is very accurately maintained, no reliable value of c can be deduced from the experiment.

130. Experiments of Bazin.—An extensive series of experiments was performed by Bazin for the purpose of determining how the flow is influenced by the character of the surface, the slope of the channel, and the size and shape of the cross-section. From these experiments the author drew the conclusion that the value of c varies but little with s, and that, so far as the dimensions of the cross-section affect it, c is mainly a function of r. The conclusions reached were embodied in a formula equivalent to the following:

$$c=c'\sqrt{\frac{r}{r+a}}.$$

The empirical constants c' and a depend upon the roughness of the channel, and Bazin gave for four typical cases the values shown in the accompanying table.

TABLE V.

VALUES OF COEFFICIENTS IN FORMULA OF BAZIN FOR OPEN CHANNELS.

Kind of Channel.	c'	a
Very smooth. (Cement or planed timber.)	143	0.10
Smooth. (Ashlar, brickwork, rough timber.)	127	0.23
Rough. (Rubble masonry.)	113	0.82
Very rough. (Ditches or canals in bad order.)	104	4.10

It will be noticed that c' is the limiting value approached by c as r increases. The range of the experiments does not, however, justify the use of the formula for values of r greater than 2 or 3 feet.

131. Kutter's Formula.—The only wholly general rule for choosing the value of c which has been widely used is the formula proposed by Ganguillet and Kutter. This formula is the result of a careful study of all the experimental data regarding flow in open channels which was known to its authors. It is designed to apply to the widest range of cases, from the smallest of artificial channels up to large rivers, and to take account of all factors affecting the value of the coefficient.

In Kutter's formula c is expressed in terms of the slope s, the hydraulic radius r, and a third quantity n called the coefficient of roughness. For English units the expression is

$$c=\frac{41.65+\dfrac{.00281}{s}+\dfrac{1.811}{n}}{1+\left(41.65+\dfrac{.00281}{s}\right)\dfrac{n}{\sqrt{r}}}.$$

The authors of the formula gave values of n for six typical kinds of channel surface, ranging from .010 for flumes lined with well-planed timber or neat cement to .030 for streams impeded by detritus or aquatic plants. The following more detailed series of values is often given:

Nature of Channel.	n
Well-planed timber	.009
Neat cement	.010
Cement one-third sand	.011
Ashlar and brickwork	.013
Canvas on frames	.015
Rubble masonry	.017
Canals in very firm gravel	.020
Rivers and canals in perfect order, free from stones or weeds	.025
Rivers and canals in moderately good order	.030
Rivers and canals in bad order, with weeds and detritus	.035
Torrential streams encumbered with detritus	.050

While these values may be fairly reliable for channels of uniform slope and alignment, greater values should probably

be used in designing flumes and ditches under ordinary practical conditions. The range of values given in Fig. 69 may be suggested.

132. Reliability of Kutter's Formula.—This formula has gained wide acceptance as a guide in choosing values of c for all cases of open channels, and also for large pipes conveying water under pressure. It is undoubtedly a formula of great value, since it provides a rule applicable to cases for which otherwise there would be no guide whatever because of the lack of experimental data. The student should, however, be warned against the too implicit acceptance of the results of this or any other empirical formula in hydraulics, and especially against the supposition that the estimated values of c (and the quantities computed from it) can be regarded as more than approximations subject to a considerable percentage of uncertainty. It is quite common for writers to give values of c to four significant figures, when in fact they can hardly be supposed to be certainly correct to within 10 per cent. Here as elsewhere in engineering computations it is important for the student to acquire the habit of putting a reasonable estimate on the degree of reliability of his data and formulas.

133. Tables Computed from Kutter's Formula.—To facilitate the use of Kutter's formula, values of c computed from it are given in Table VI. Values not given directly in the table may be found with sufficient accuracy by interpolation.

134. Graphical Representation of Kutter's Formula.—The formula of Ganguillet and Kutter may be written in the form

$$c=\frac{y}{1+\frac{x}{\sqrt{r}}}=\frac{y\sqrt{r}}{\sqrt{r}+x}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which

$$x=\left(41.65+\frac{.00281}{s}\right)n, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$y=41.65+\frac{.00281}{s}+\frac{1.811}{n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

TABLE VI.

VALUES OF c IN THE FORMULA $v = c\sqrt{rs}$, COMPUTED FROM FORMULA OF GANGUILLET AND KUTTER.

s	n	r (feet). .2	.3	.4	.6	.8	1.0	1.5	2.0	3	4	6	8	10	15
.00005	.010	87	99	109	122	133	140	154	164	178	187	199	206	212	222
	.012	68	79	88	98	107	114	126	135	148	156	168	175	181	190
	.015	51	59	66	76	83	89	96	107	118	126	137	144	149	158
	.020	35	41	46	53	59	64	72	79	88	95	105	111	116	125
	.025	26	31	35	41	46	49	57	62	71	77	85	91	96	104
	.030	21	25	28	33	37	40	47	51	59	64	72	78	82	90
	.035	18	21	24	28	31	34	40	44	50	56	63	68	72	80
	.040	15	18	20	24	27	29	34	38	44	49	56	61	64	72
.0001	.010	98	109	119	131	140	147	159	168	178	186	195	201	205	212
	.012	76	87	95	105	114	120	130	138	149	155	164	170	174	181
	.015	57	65	72	81	88	93	103	109	119	125	134	139	143	150
	.020	39	45	50	57	63	67	75	81	89	94	102	107	111	118
	.025	29	34	38	44	48	52	59	64	71	76	84	88	92	98
	.030	23	27	31	35	39	42	48	53	59	64	71	75	78	85
	.035	19	22	25	30	33	35	41	45	51	55	61	66	69	75
	.040	16	19	22	25	28	31	35	39	45	49	54	59	62	68
.0002	.010	105	116	125	138	145	151	162	170	179	185	193	198	201	207
	.012	83	92	100	111	118	123	133	140	149	155	162	167	170	176
	.015	61	69	76	85	91	96	105	111	119	125	132	137	140	145
	.020	42	48	53	60	65	69	77	82	89	94	100	105	108	113
	.025	31	36	40	46	50	54	60	64	72	76	82	87	89	95
	.030	25	29	32	37	41	44	49	54	59	63	69	73	76	82
	.035	21	24	27	31	34	37	42	45	51	55	60	64	67	72
	.040	17	20	23	26	29	32	36	40	45	48	53	57	60	65
.0004	.010	110	120	129	140	148	154	164	170	179	184	191	196	199	204
	.012	87	96	104	113	121	125	135	141	149	154	161	165	168	174
	.015	65	73	79	87	93	98	106	112	119	124	130	135	138	143
	.020	44	50	55	62	67	70	78	83	89	94	99	104	107	112
	.025	32	37	42	47	51	55	60	65	71	76	81	85	88	94
	.030	25	30	33	38	42	45	50	54	59	63	69	73	75	81
	.035	21	24	27	31	35	37	42	45	51	55	60	64	66	72
	.040	18	21	23	27	30	32	37	40	45	48	53	57	59	65
.0010	.010	113	124	131	142	150	155	165	171	179	184	190	194	197	202
	.012	89	98	105	115	122	127	136	142	149	154	160	164	167	172
	.015	66	74	80	88	94	99	108	112	119	124	130	134	136	141
	.020	45	51	56	63	68	71	78	83	89	93	99	103	105	110
	.025	34	39	43	48	52	56	62	66	71	75	81	85	87	92
	.030	27	30	34	39	42	45	50	54	59	63	68	72	74	79
	.035	22	25	28	32	35	38	43	46	51	54	59	63	65	70
	.040	18	21	24	27	30	33	37	40	45	48	52	56	58	63

If s be eliminated between these equations, there results an equation between x, y, and n. For a given constant value of n this equation may be represented by a curve with x and y as rectangular coordinates. A series of such curves may thus be drawn, each corresponding to a definite value of n. In like manner a second series of curves may be determined, each corresponding to a definite value of s.

Thus, eliminating s,

$$x=ny-1.811. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Eliminating n,

$$x\left[y-\left(41.65+\frac{.00281}{s}\right)\right]=1.811\left(41.65+\frac{.00281}{s}\right). \quad . \quad (5)$$

Taking rectangular axes OX, OY (Fig. 68), let equation (4) be plotted for some definite value of n. The resulting locus is a straight line AB, such that $OA=1.811$, $OB=\frac{1.811}{n}$. For a second value of n another straight line is obtained passing through the same point A.

Next consider the curve represented by (5) when s has any constant value. This is seen to be a rectangular hyperbola

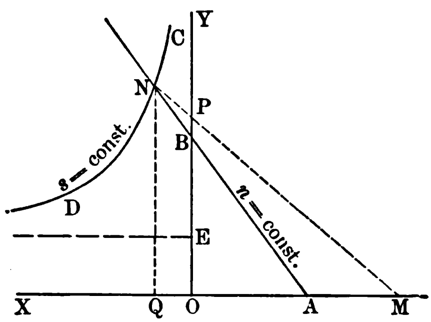


FIG. 68.

whose asymptotes are OY and a line parallel to OX at a distance from it $OE=41.65+\frac{.00281}{s}$.

Let lines such as AB be drawn for a series of values of n, and curves such as CD for a series of values of s. The point

in the plane OXY which corresponds to any definite values of n and s can then be located by inspection. The values of x and y are thus known, and c may be found by the following graphical construction.

For any given value of r, let OM (Fig. 68) be taken equal to $\sqrt{r}$. Let N be the point whose co-ordinates are x, y, determined from any given values of n and s. Draw MN, and let P be its point of intersection with OY. From similar triangles,

$$\frac{OP}{QN}=\frac{OM}{QM},$$

or
$$OP=\frac{OM\times QN}{QM}=\frac{y\sqrt{r}}{\sqrt{r}+x}.$$

Comparing with (1),
$$OP=c.$$

From such a diagram it is possible not only to determine c when r, s, and n are given, but to determine any one of the four quantities when the other three are given. For accurate computations the diagram should be drawn carefully to a large scale. But even the diagram shown in Fig. 69 suffices for making computations with less uncertainty than that of the data usually involved in practical hydraulic problems.*

135. Smallest Cross-section for Given Rate of Discharge. —If, with a given slope, a channel is to discharge water at a given rate, the cross-sectional area will depend upon the form of section adopted. Let it be required to make this area as small as possible.

In the equation

$$q=Fv=Fc\sqrt{rs}=c\sqrt{s}\frac{F^{\frac{3}{2}}}{C^{\frac{1}{2}}}$$

the conditions of the problem make s constant, and it will be assumed that c is constant. Then q being expressed as a function of two variables F and C, it is required to satisfy the conditions

$$q=\text{constant},\qquad F=\text{a minimum}.$$

* The computation of velocity for any values of r s, and n is greatly facilitated by the use of diagrams published by Prof. I. P. Church.

DIAGRAM BASED ON FORMULA OF GANGUILLET AND KUTTER.

$$c=\frac{41.65+\frac{.00281}{s}+\frac{1.811}{n}}{1+\left(41.65+\frac{.00281}{s}\right)\frac{n}{\sqrt{r}}}.$$

Character of Channel.	Value of n.
Planed timber or neat cement. . .	.010 –.012
Cement mortar or rough timber.	.012 –.014
Ashlar or brickwork.	.013 –.015
Rubble masonry or firm gravel. .	.017 –.022
Ditches in clean earth.	.025 –.030
Canals and rivers.	.030 –.050

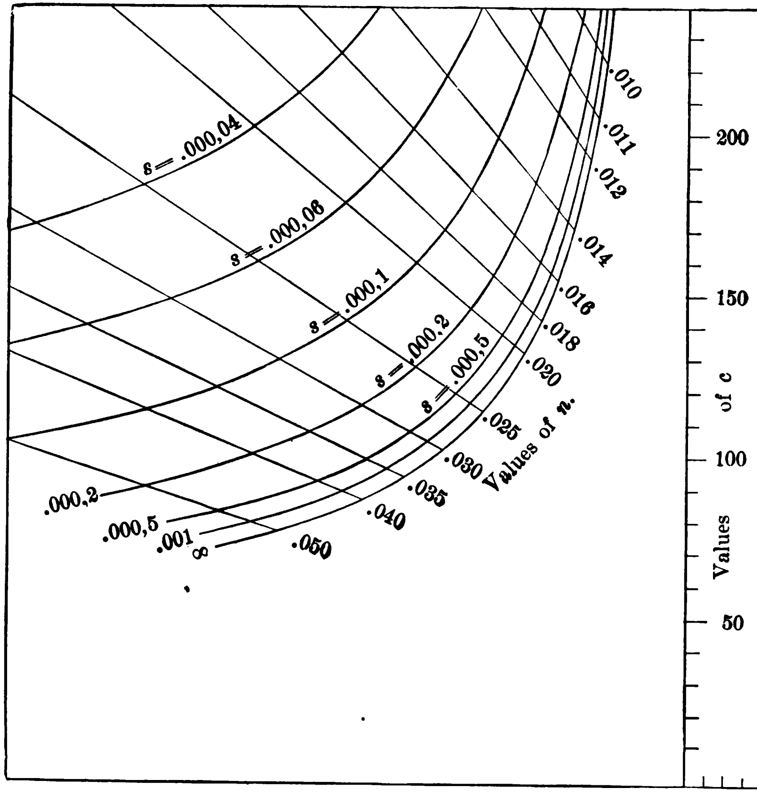


FIG. 69.

From the form of the above value of q it follows that, q being constant, C is a minimum when F is.

The values of F and C admit of infinite variation, depending upon the form chosen for the cross-section. The discussion will be restricted to the case of a trapezoidal section, represented in Fig. 70.

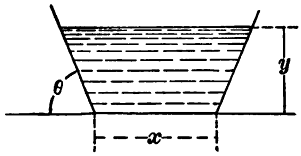


FIG. 70.

Trapezoidal section. — Let $x=$ bottom width, $y=$depth of water, $\theta=$angle between side slope and horizontal; then F and C can be expressed in terms of x and y. For any fixed value of θ let it be required to determine what relation between x and y will give minimum values of F and p.

We have
$$F=y(x+y \text{ cotan } \theta);$$
$$C=x+2y \text{ cosec } \theta.$$

The conditions to be satisfied are
$$dF=0, \qquad dC=0.$$

Differentiating,
$$dF=y\,dx+(x+2y \text{ cotan } \theta)dy=0;$$
$$dC=dx+2 \text{ cosec } \theta \cdot dy=0.$$

Eliminating dy/dx between these equations,
$$x=\frac{2(1-\cos\theta)}{\sin\theta}y,$$

which is the relation between x and y for minimum cross-section and wetted perimeter. From this relation are deduced the following:
$$F=y(x+y \text{ cotan } \theta)=(2 \text{ cosec } \theta-\text{cotan } \theta)y^2;$$
$$C=(x+2y \text{ cosec } \theta)=2(2 \text{ cosec } \theta-\text{cotan } \theta)y;$$
$$r=\frac{F}{C}=\frac{y}{2};$$
$$v=c\left(\frac{s}{2}\right)^{\frac{1}{2}} y^{\frac{1}{2}};$$
$$q=(2 \text{ cosec } \theta-\text{cotan } \theta)\,c\left(\frac{s}{2}\right)^{\frac{1}{2}} y^{\frac{5}{2}}.$$

Rectangular section.—If $\theta=90°$, the solution for minimum cross-section and wetted perimeter becomes

$$x=2y; \qquad F=2y^2; \qquad C=4y; \qquad r=\frac{y}{2};$$

$$v=c\left(\frac{s}{2}\right)^{\frac{1}{2}}y^{\frac{1}{2}};$$

$$q=c(2s)^{\frac{1}{2}}y^{\frac{5}{2}}.$$

EXAMPLES.

In the following examples let c be determined by Kutter's formula.

1. A rectangular flume of rough plank is to be laid on a slope of 1/1000, and is to discharge 10 cu. ft. per sec. Estimate the width and depth for minimum cross-section and wetted perimeter.

Ans. $y=1.3'$, $x=2.6'$.

2. A ditch with trapezoidal section, side slopes 45°, slope of bed 1/1000, is to discharge 40 cu. ft. per sec. Required the best values of the width and depth. Assume $n=.025$. *Ans.* $y=3.0'$.

3. Compute the rate of discharge of a rectangular flume of brick masonry, 3.5 ft. wide, sloping 1 ft. in 3000 ft., the depth of the water being 2 ft. *Ans.* About 13.6 cu. ft. per sec.

4. Prove that, of all trapezoidal sections giving a certain rate of discharge, the section of minimum area has side slopes of 60°.

CHAPTER XII.

OPEN CHANNELS: NON-UNIFORM FLOW.

136. General Equation of Energy for Stream of Variable Cross-section.—The reasoning by which the general equation

$$z_1+\frac{p_1}{w}+\frac{v_1^2}{2g}=z_2+\frac{p_2}{w}+\frac{v_2^2}{2g}+H'$$

was established (Arts. 59, 60) is valid in the case of a stream with a free upper surface. For such a stream the hydraulic gradient coincides with the surface of the stream, and $z+p/w$ or y denotes the height of the water surface above a horizontal datum plane. (See Fig. 67.)

In Fig. 71 let A and B be any two cross-sections of a stream, the flow being from A toward B; let y_1, v_1 be the values of y and v at A, and y_2, v_2 the values at B; and let H' denote the loss of head between A and B (i.e., the energy lost between A and B for every pound of water passing any cross-section of the stream). Then

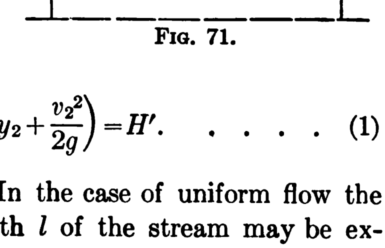


FIG. 71.

$$\left(y_1+\frac{v_1^2}{2g}\right)-\left(y_2+\frac{v_2^2}{2g}\right)=H'. \quad . \quad . \quad . \quad . \quad (1)$$

137. Value of Lost Head.—In the case of uniform flow the loss of head in any given length l of the stream may be expressed by means of the formula

$$v=c\sqrt{rs}.$$

For in this case, v_1 and v_2 being equal,

$$H' = y_1 - y_2$$

and
$$s = \frac{y_1 - y_2}{l} = \frac{H'}{l},$$

so that
$$H' = ls = \frac{lv^2}{c^2 r}. \qquad (2)$$

Let this same expression for H' be assumed for the case of variable flow, it being understood that v and r have values intermediate between those applying to the sections A and B. Equation (1) then becomes

$$\left(y_1 + \frac{v_1^2}{2g}\right) - \left(y_2 + \frac{v_2^2}{2g}\right) = \frac{lv^2}{c^2 r}. \qquad (3)$$

138. Differential Equation for Surface Curve.—Consider a longitudinal section of the stream by a vertical plane, and let x, y be the coordinates of the surface curve referred to axes OX, OY, the former being horizontal and the latter vertical

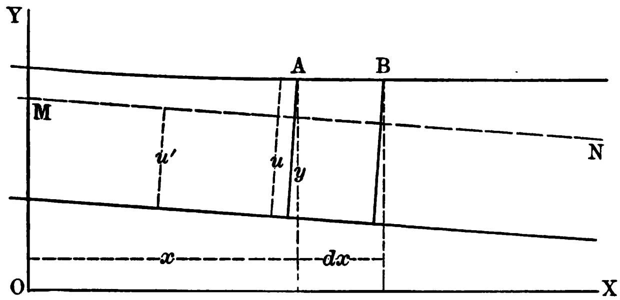


FIG. 72.

(Fig. 72). If the above equation be applied to the portion of the stream between two sections A and B infinitely near together, we must put

$$l = dx, \quad y_2 - y_1 = dy,$$
$$v_2^2 - v_1^2 = d(v^2) = 2v\,dv,$$

so that the equation becomes

$$dy+\frac{v}{g}dv+\frac{v^2}{c^2r}dx=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

139. Channel of Uniform Shape and Slope.—Let the channel have a uniform slope and uniform cross-section, the cross-section of the stream, however, varying with the depth of the water. Let

i = slope of bed;
u = depth of stream at point (x, y);
b = width of stream at water surface.

Let y be replaced by u, by means of the evident relation

$$dy=du-i\,dx.$$

Also, we have

$$v=\frac{q}{F} \text{ (q being constant)};$$

$$dF=b\,du;$$

$$dv=-\frac{q\,dF}{F^2}=-\frac{qb\,du}{F^2}$$

Equation (4) may therefore be written

$$\left(i-\frac{q^2}{c^2rF^2}\right)dx=\left(1-\frac{q^2b}{gF^3}\right)du. \quad . \quad . \quad . \quad . \quad (5)$$

Now let q be replaced by its value in terms of the dimensions of a uniform stream in the same channel. Thus, if the channel were unobstructed for a great distance, the surface would assume some position as MN (Fig. 72) parallel to the bed. Let u', F', r' be the values which u, F, r would have in such a case; then

$$q=F'c\sqrt{r'i},$$

and equation (5) takes the form

$$\left(1-\frac{r'F'^2}{rF^2}\right)idx=\left(1-\frac{c^2i}{g}\cdot\frac{br'F'^2}{F^3}\right)du. \quad . \quad . \quad . \quad . \quad (6)$$

When the form of the channel is given, F, b and r can be expressed in terms of u, and the relation between u and x is then to be found by integrating equation (6). In certain ideal cases the integration is fairly simple. The one most commonly treated is that in which the cross-section is assumed to be of uniform depth and of great width.

140. Cross-section of Great Width and Uniform Depth.—For a rectangular cross-section b is constant and

$$F = bu, \qquad r = \frac{bu}{b+2u};$$

$$F' = bu', \qquad r' = \frac{bu'}{b+2u'},$$

u' being the depth corresponding to cross-section F'.

If b is very great in comparison with u, we have approximately

$$r = u, \quad r' = u',$$

and equation (6) becomes

$$\left(\frac{u^3}{u'^3} - 1\right) i\, dx = \left(\frac{u^3}{u'^3} - \frac{c^2 i}{g}\right) du. \quad . \quad . \quad . \quad . \quad (7)$$

Let $u/u' = z$; then $du = u'dz$, and

$$i(z^3 - 1)dx = \left(z^3 - \frac{c^2 i}{g}\right) u'dz,$$

or

$$\frac{i}{u'}dx = dz + \left(1 - \frac{c^2 i}{g}\right)\frac{dz}{z^3 - 1}. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

For brevity let

$$-\int \frac{dz}{z^3 - 1} = \phi(z);$$

then the integration of (8) gives

$$\frac{ix}{u'} = z - \left(1 - \frac{c^2 i}{g}\right)\phi(z) + \text{constant}. \quad . \quad . \quad . \quad . \quad (9)$$

Let the integration be taken between limits $x=x_1$ and $x=x_2$, where $x_2-x_1=l=$distance between any two definite cross-sections of the stream, z_1, z_2 being the corresponding values of z. Then

$$\frac{li}{u'}=z_2-z_1-\left(1-\frac{c^2i}{g}\right)[\phi(z_2)-\phi(z_1)]. \quad . \quad . \quad . \ (10)$$

In order to apply this equation it is necessary to compute values of $\phi(z)$ for a series of values of z. The general value of the function is found by integration * to be

$$\phi(z)=-\int\frac{dz}{z^3-1}=\frac{1}{6}\log\frac{z^2+z+1}{(z-1)^2}-\frac{1}{\sqrt{3}}\cot^{-1}\left(\frac{2z+1}{\sqrt{3}}\right).$$

From this are computed the values entered in Table VII.

For very exact computations a fuller table is required. Intermediate values may, however, be found by interpolation with less error than that due to the defects in the above theory and the uncertainty always existing in the data of hydraulic problems.

TABLE VII.

VALUES OF $\phi(z)=-\int\frac{dz}{z^3-1}$.

z	$\phi(z)$	z	$\phi(z)$	z	$\phi(z)$	z	$\phi(z)$
1.00	∞	1.10	.680	1.30	.373	1.65	.203
1.01	1.419	1.12	.626	1.32	.357	1.70	.189
1.02	1.191	1.14	.581	1.34	.342	1.80	.166
1.03	1.060	1.16	.542	1.36	.328	1.90	.147
1.04	.967	1.18	.509	1.38	.316	2.00	.1318
1.05	.896	1.20	.480	1.40	.304	2.10	.1188
1.06	.838	1.22	.454	1.45	.278	2.20	.1074
1.07	.790	1.24	.431	1.50	.255	2.30	.0978
1.08	.749	1.26	.410	1.55	.235	2.40	.0894
1.09	.713	1.28	.390	1.60	.218	2.50	.0822

141. Backwater.—If the surface of a stream at a certain point be raised by a dam or other obstruction, the effect will extend up-stream for some distance, decreasing as the distance

* See Williamson's Integral Calculus, p. 59. To the value there given is added such a constant that $\phi(z)=0$ when $z=\infty$.

from the obstruction increases. To estimate the amount by which the surface is raised, at a given distance above the dam, is a practical question often submitted to the hydraulic engineer. If the stream is broad and shallow, and if the bed has a fairly uniform slope, the foregoing theory may be applied. The method of procedure may be illustrated by the following example.

Suppose a stream is originally 20 ft. deep, and that the surface is raised by an obstruction so that the depth at a certain point becomes 30 ft. Let the slope of the bed be 1/5000 and let $c=80$. It is required to determine at what distance from the given section the increase in depth is 2 ft.

From the given data $u'=20$, $i=.0002$, $c=80$, $z_2=1.5$, $z_1=1.1$. The table gives $\phi(z_1)=.680$, $\phi(z_2)=.255$; hence

$$l=100000[(1.5-1.1)+.960(.680-.255)]=80800 \text{ ft.}$$

If, instead of z_1, l is given and z_1 is required, the solution is not so direct. Thus, suppose in the above case it is required to determine the increase in depth at a section 25000 ft. above the point where the depth is 30 ft.; that is, $l=25000$ and z_1 is required. Substituting in equation (10),

$$z_1-.960\phi(z_1)=1.005.$$

By taking trial values from the table it is found that the equation is nearly satisfied by $z_1=1.34$. The depth at the specified section is therefore 1.34×20 ft. $=26.8$ ft.

In applying the above theory to actual streams for which the assumption of uniform depth throughout the cross-section is not even roughly true, u may be taken as the average depth for the cross-section, computed from the formula $u=F/b$. This method may be applied to Ex. 4 of the following list.

EXAMPLES.

1. A wide stream 12 ft. deep is obstructed so that the depth at a certain section becomes 16 ft. The slope of the bed is 2 ft. per mile. Compute the depth (*a*) 2 miles up-stream and (*b*) 2 miles down-stream. [Take $c = 65$.] *Ans.* (*a*) 14.0 ft. (*b*) 18.7 ft.

2. In Ex. 1, determine the positions of sections at which the depth has values 13 ft., 15 ft., and 16 ft.
Ans. The depth would be 13 ft. at about 3.73 miles up-stream.

3. A wide stream 20 ft. deep is obstructed so that the depth at a certain section becomes 25 ft. If $i = .0002$ and $c = 70$, determine (*a*) the depth 10000 ft. up-stream from the given section, and (*b*) where the depth will be 21 ft. *Ans.* (*a*) 24.1 ft. (*b*) About 66000 ft. up-stream.

4. A stream whose cross-sectional area is 5400 sq. ft. and width 350 ft. when unobstructed has its surface raised 10 ft. at a certain section by a dam. The slope is 1.6 ft. per mile, and the value of c is 75. (*a*) Compute the rise of the surface at distances of 2 miles and 5 miles above the given section. (*b*) Where will the increase of depth be 5 ft.?
Ans. (*a*) 7.6 ft. and 4.6 ft. (*b*) At 4.53 miles up-stream.

CHAPTER XIII.

THE MEASUREMENT OF RATE OF DISCHARGE.

142. General Methods.—One of the most important of the problems of practical Hydraulics is the determination of the rate of discharge of streams. The conditions of the problem vary greatly in different cases, and the methods employed must vary correspondingly. The quantities to be measured vary from the discharge of a small pipe to that of a large river.

The methods employed may be classed as (1) direct and (2) indirect.

(1) An actual measurement may be made of either (*a*) the total quantity discharged in a given time, or (*b*) the velocity of flow.

(2) Measurement may be made of certain quantities upon which the rate of discharge is known to depend, its value being computed from the data thus determined.

143. Direct Measurement of Total Discharge.—The total quantity discharged in a given time may be determined either by weighing the water or by measuring its volume in a properly calibrated vessel or reservoir.

Accurate measurement by weighing will in general be possible only when the rate of discharge is small, since for accurate results the experiment must extend over a considerable time. A discharge of 1 cu. ft. per second would give in 1 minute a total discharge of 3750 lbs.

If a reservoir is available whose volume for given depths is accurately known, the total discharge in a given time may be measured by collecting the water in this reservoir. Much larger quantities can thus be measured than it is practicable to weigh.

This method is sometimes employed in estimating the rate of discharge of the conduit of a water supply system, the discharge for a period of several hours or even a day or more being collected in a large storage reservoir whose horizontal area at different levels is accurately known. Such measurements are liable to uncertainty because of evaporation and leakage, and also because a small error in measuring the depth of the water results in a relatively large error in the total quantity.

144. Discharge Computed from Direct Measurement of Velocity.—If the velocity of flow be measured at many different points in a given cross-section of a stream, the rate of discharge for the whole section may be closely estimated. Let the total cross-section be F, and let it be divided into parts whose areas are $F_1, F_2, \ldots$, the values of the mean velocity in these parts being $v_1, v_2, \ldots$, and the mean velocity for the entire section v. Then the total rate of discharge is

$$q = Fv = F_1v_1 + F_2v_2 + \ldots$$

This method is especially applicable to open streams of considerable size.

145. Methods of Measuring Velocity.—Of the various devices that have been employed for measuring velocity of flow there may be mentioned floats, the tachometer or current-meter, and the Pitot tube.

Floats are especially applicable to streams of considerable size. Under favorable conditions they may be used for measuring the velocity not only at the surface but at any desired depth, being suspended by wires from surface floats of such shape and size as to be little influenced by the surface velocity. Floats are also sometimes made in the form of tubes so weighted as to float in an upright position and extend from the surface nearly to the bottom, the motion of a float thus indicating a mean of the velocities at different depths. The field operations required in this and other methods of gauging the flow of large streams are quite elaborate and will not be here described.*

* Consult Physics and Hydraulics of the Mississippi River, by Humphreys and Abbot.

The current-meter consists of a vaned wheel which is rotated by the current at a rate depending upon the velocity. The total number of revolutions in a given time is counted, usually by means of an automatic register. The relation between the velocity of flow and the number of revolutions per unit time must be determined for any given instrument by experiment. This is called "rating" the meter.

Pitot's tube, in its simplest form, is an open tube one end of which is bent at right angles to the length of the main portion. The bent end is submerged and placed with the opening facing the current, while the other end projects above the free surface (Fig. 73). The water in the tube rises above the surface of the stream to a height depending upon the velocity of flow at the lower end. No exact relation between the velocity v and the height h of the column above the water surface can be determined from theory alone. It would seem that the increase of pressure at the lower end of the tube will be proportional approximately to the square of the velocity, so that

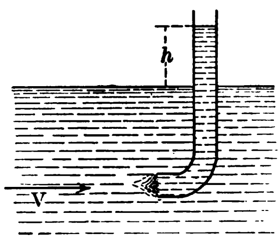


FIG. 73.

$$h = k\frac{v^2}{2g},$$

k being a coefficient whose value for any given instrument must be determined experimentally. (See Art. 171.)

As constructed for practical use, the instrument usually consists of two tubes whose lower ends are open and are directed at an angle of 90° or 180° with each other, while the upper ends are open either to the atmosphere or to a common pressure chamber, or else communicate with a gauge for reading their pressure-difference. Such an instrument has been successfully used both with open channels and with pipes under pressure.

146. Indirect Methods of Measuring Rate of Discharge.—Of the methods which above were called indirect, the most important are three: by orifices, by weirs, and by the Venturi meter. These will be considered in order.

147. Measurement by Orifices.—If a stream be conducted into a tank or reservoir in the side of which is an orifice, the surface of the water will assume such a position that the rate of discharge through the orifice becomes equal to the rate at which water flows into the reservoir.

The rate of discharge of an orifice of cross-section F, under a head h on its center, is

$$q=cF\sqrt{2gh}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

c being the coefficient of discharge (Art. 44). This formula may be used unless the dimensions of the orifice are relatively large in comparison with the head. For a large orifice whose plane is not horizontal the form of the formula depends upon the shape and cross-section of the orifice, as shown in Art. 46. It is rarely needful, however, to employ the theoretically more accurate formula, since the difference between the results given by it and by the approximate formula is usually much less than the uncertainty in the value of the coefficient of discharge. Thus for a circular orifice in a vertical plane the accurate formula may be developed in a series of which the first two terms are as follows:

$$q=cF\sqrt{2gh}\left[1-\frac{1}{128}\left(\frac{d}{h}\right)^2-\ldots\right], \quad . \quad . \quad . \quad . \quad (2)$$

d being the diameter. If $d/h=1$, the second term is less than one per cent of the first, and the series converges rapidly. For a rectangular orifice of width b the exact formula (Art. 49) is

$$q=c\tfrac{2}{3}\sqrt{2g}\,b(h_2^{\frac{3}{2}}-h_1^{\frac{3}{2}}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which h_1 and h_2 are the depths of the upper and lower edges below the water surface. Putting $h_2=h+d/2$ and $h_1=h-d/2$, this can be expressed as a series of which two terms are as follows:

$$q=cF\sqrt{2gh}\left[1-\frac{1}{96}\left(\frac{d}{h}\right)^2-\ldots\right]. \quad . \quad . \quad . \quad (4)$$

If $d/h=1$, the second term is a little greater than one per cent of the first, and the series converges quite rapidly.

The coefficients of discharge for orifices made in a standard way are fairly well known by experiment. The accompanying tables * give values of c for circular standard orifices, square orifices, and rectangular orifices one foot wide. In every case the orifice is supposed to be formed with a sharp inner edge (as at A, Fig. 17), so that full contraction of the jet occurs. It is also needful that the orifice shall be at a considerable distance from the sides and bottom of the reservoir; otherwise the approaching particles of water will be so guided that full contraction will be prevented and the value of c will be uncertain.

The horizontal lines in the tables indicate the limiting values of h below which the approximate formula (1) should be replaced by (2) or (3).

TABLE VIII.

VALUES OF COEFFICIENT OF DISCHARGE FOR CIRCULAR VERTICAL ORIFICES

h (feet).	d (feet).						
	0.02	0.04	0.07	0.1	0.2	0.6	1.0
0.4		0.637	0.624	0.618			
0.6	0.655	.630	.618	.613	0.601	0.593	
0.8	.648	.626	.615	.610	.601	.594	0.590
1.	.644	.622	.612	.608	.600	.595	.591
1.5	.637	.618	.608	.605	.600	.596	.593
2.	.632	.614	.607	.604	.599	.597	.595
2.	.629	.612	.605	.603	.599	.598	.596
3.5	.627	.611	.604	.603	.599	.598	.597
4.	.623	.607	.603	.602	.599	.597	.596
6.	.618	.607	.602	.600	.598	.597	.596
8.	.614	.605	.601	.600	.598	.596	.596
10.	.611	.603	.599	.598	.597	.596	.595
20.	.601	.599	.597	.596	.596	.596	.594
50.	.596	.595	.594	.594	.594	.594	.593
100.	.593	.592	.592	.592	.592	.592	.592

* These tables are taken from Hamilton Smith's Hydraulics

TABLE IX.

VALUES OF COEFFICIENT OF DISCHARGE FOR SQUARE VERTICAL ORIFICES

h (feet).	d (feet).						
	0.02	0.04	0.07	0.1	0.2	0.6	1.0
0.4		0.643	0.628	0.621			
0.6	0.660	.636	.623	.617	0.605	0.598	
0.8	.652	.631	.620	.615	.605	.600	0.597
1.	.648	.628	.618	.613	.605	.601	.599
1.5	.641	.622	.614	.610	.605	.602	.601
2.	.637	.619	.613	.608	.605	.604	.602
2.5	.634	.617	.610	.607	.605	.604	.602
3.	.632	.616	.609	.607	.605	.604	.603
5.	.628	.614	.608	.606	.605	.603	.602
6.	.623	.612	.607	.605	.604	.603	.602
8.	.619	.610	.606	.605	.604	.603	.602
10.	.616	.608	.605	.604	.603	.602	.601
20.	.606	.604	.602	.602	.602	.601	.600
50.	.602	.601	.601	.600	.599	.600	.599
100.	.599	.598	.598	.598	.598	.598	.598

TABLE X.

VALUES OF COEFFICIENT OF DISCHARGE FOR RECTANGULAR ORIFICES 1 FT. WIDE.

h (feet).	d (feet).						
	0.125	0.25	0.50	0.75	1.0	1.5	2.0
0.4	0.634	0.633	0.622				
0.6	.633	.633	.619	0.614			
0.8	.633	.633	.618	.612	0.608		
1.	.632	.632	.618	.612	.606	0.626	
1.5	.630	.631	.618	.611	.605	.626	0.628
2.	.629	.620	.617	.611	.605	.624	.630
2.5	.628	.628	.616	.611	.605	.616	.627
3.	.627	.627	.615	.610	.605	.614	.619
4.	.624	.624	.614	.609	.605	.612	.616
6.	.615	.615	.609	.604	.602	.606	.610
8.	.609	.607	.603	.602	.601	.602	.604
10.	.606	.603	.601	.601	.601	.601	.602
20.				.601	.601	.601	.602

148. Miner's Inch.—Where water is sold for hydraulic mining or for irrigation the unit of measurement commonly employed is the miner's inch. This is usually understood to mean the discharge through an orifice one inch square under some specified head; multiples of the "inch" being obtained by increasing the horizontal dimension of the orifice while leaving the head unchanged. Various definitions have, however, been accepted at different times and in different localities. In California a common definition of the miner's inch has been the discharge from an orifice one inch square when the head on the upper edge is four inches, and hydraulic engineers have quite generally accepted 1.2 cubic feet per minute as its equivalent, corresponding to a value of about 0.59 for c. The legal equivalent* is, however, 1.5 cubic feet per minute.

149. Measurement by Weirs.—A weir is a notch in the side of a vessel or reservoir. Such notches are used for the measurement of rate of discharge, and it is essential that their construction shall conform to accepted standards in order that reliable results may be obtained.

The most common form of weir is rectangular. The standard rectangular weir is constructed with the lower edge or "crest" sharp (like the edge of a standard orifice), so as to permit full "crest contraction" of the stream. The vertical edges should either be sharp or else should lie in the vertical sides of the channel of approach. In the former case there is full "end contraction" of the stream; in the latter there will be no lateral contraction at the ends, which is commonly expressed by saying that the end contractions are "suppressed."

It is essential also that the depth of the channel of approach shall be sufficient so that full crest contraction is not interfered with. Also, unless the end contractions are completely suppressed, the channel of approach should be considerably wider than the length of the weir, in order that full contraction may occur at the ends.

* Since 1901.

The formula for the rate of discharge over a rectangular weir is given in Art. 50. It is

$$q=c\tfrac{2}{3}\sqrt{2g}\,bH^{\frac{3}{2}}, \quad . \quad . \quad . \quad (5)$$

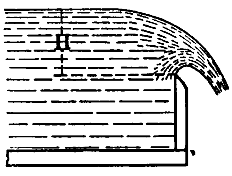


FIG. 74.

c being a coefficient whose value must be determined by experiment.

The "head on the crest" (H) must be measured to the level of the surface of still water back of the weir (Fig. 74).

150. The Francis Formula.—Elaborate experiments were made by J. B. Francis * for the purpose of determining the value of the coefficient c in the above formula, and also the effect of end contractions. The weirs used in these investigations ranged in length from 4 ft. to 10 ft., and H ranged from 0.6 ft. to 1.6 ft. As a result of these experiments their author adopted an average value 0.622 for the coefficient c. His formula, for the case of suppressed end contractions, is

$$q=3.33bH^{\frac{3}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

b and H being in feet.

As regards end contractions, Francis concluded that each contraction decreases the effective length of the weir by an amount proportional to H, and gave the formula

$$q=3.33(b-0.1nH)H^{\frac{3}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

in which n is the number of end contractions, usually two.

Velocity of approach.—If the channel of approach is so small that the velocity of the approaching water is appreciable, the formula of Art. 68 is taken as the basis, giving the following:

$$q=3.33(b-0.1nH)[(H+h')^{\frac{3}{2}}-h'^{\frac{3}{2}}], \quad . \quad . \quad . \quad . \quad (8)$$

in which $h'=\dfrac{v'^2}{2g}$, v' being the velocity of approach. If v' denotes the average velocity in the cross-section of the channel, h' should probably be taken greater than $\dfrac{v'^2}{2g}$, since the

* Lowell Hydraulic Experiments.

surface velocity is generally greater than the mean velocity. Different authors have used values ranging from $\frac{v'^2}{2g}$ to $2\frac{v'^2}{2g}$. If $\frac{h'}{H}$ is a small fraction the term $h'^{\frac{3}{2}}$ may be neglected in comparison with $(H+h')^{\frac{3}{2}}$. This simplification of formula (8) is often used.

For weirs with full crest and end contractions the formula of Francis is probably as reliable as any available rule, especially for values of H ranging from .6 ft. to 1.6 ft. The discharge of weirs with end contractions suppressed has, however, been more fully investigated by Bazin and others.

151. Experiments and Formula of Bazin.—Careful experiments were made by Bazin to determine the discharge over sharp-crested weirs without end contractions and with free admission of air below the overfalling stream. The weirs used were of three lengths, 2 m., 1 m., and .5 m., and the depth of the channel of approach below the crest varied from .240 m. to 1.135 m. His results are embodied in Table XI, giving values of the coefficient * m in the formula

$$q = mbH\sqrt{2gH} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

for different values of the head on the crest (H) and of the height (G) of the crest above the bottom of the channel of approach. The variation of m with G thus shows the effect of velocity of approach. In the last column of the table are given values of m corresponding to no velocity of approach; these are inferred rather than obtained by direct experiment. No variation of m with the length of the weir was shown by the experiments.

Formula of Bazin.—The formula adopted by Bazin was

$$m = \mu\left[1 + .55\left(\frac{H}{G+H}\right)^2\right], \quad . \quad . \quad . \quad . \quad (10)$$

the form of which was suggested by theory,† while the numerical coefficient was determined from a discussion of the experi-

* It will be observed that m here replaces $\frac{2}{3}c$ in formula (5), p. 149.

† See Annales des Ponts et Chaussées, t. XVI (1888).

ments. He also gave for μ (the value of m corresponding to no velocity of approach) a formula equivalent to

$$\mu = .405 + \frac{.00984}{H}, \quad \ldots \ldots \quad (11)$$

H being in feet. Within the range of the experiments (shown approximately by Table XI), the values of m computed from the formula are within 2% of those obtained experimentally; for $G > 1$ ft. the agreement is within 1%.

TABLE XI.

VALUES OF m IN FORMULA $q = mH\sqrt{2gH}$, FOR WEIRS WITHOUT END CONTRACTIONS. (From Experiments of Bazin.)

H (feet)	G (feet)							
	.75	1.00	1.25	1.50	2.00	3.00	4.00	∞ ($m=\mu$)
.2	.445	.445	.445	.445	.445	.445	.445	.4423
.3	.447	.441	.439	.438	.437	.436	.435	.4337
.4	.450	.443	.439	.436	.434	.432	.430	.4289
.5	.454	.447	.442	.437	.434	.431	.428	.4254
.6	.460	.452	.445	.440	.435	.430	.426	.4227
.7	.466	.457	.448	.443	.436	.430	.425	.4207
.8	.473	.462	.452	.446	.439	.430	.424	.4193
.9	.479	.467	.456	.449	.441	.431	.424	.4183
1.0	.485	.472	.460	.452	.443	.432	.424	.4173
1.1	.492	.477	.464	.455	.446	.433	.425	.4163
1.2	.499	.483	.469	.459	.448	.434	.425	.4154
1.3	.506	.488	.474	.463	.450	.435	.426	.4145
1.4	.513	.494	.479	.467	.453	.436	.426	.4137
1.5	.520	.500	.484	.470	.455	.438	.427	.4130
1.6				.474	.458	.439	.427	.4120
1.7						.441	.428	.4113
1.8						.442	.429	.4104
1.9						.444	.429	.4096
2.0						.445	.430	.4092

For values of H much beyond the range of Bazin's experiments his formula is probably not more reliable than the simpler one of Francis. For $H < .2$ ft. Bazin's formula should not be used.

If the channel of approach is of irregular section, formula

(10) can be applied only by estimating the value of G for an assumed "equivalent" uniform channel.*

End contractions.—For weirs with full end contractions Bazin's coefficients may be used in connection with the Francis correction for contraction; i.e., in formula (9) b may be replaced by $b-.2H$.

152. Weirs Not of Standard Form.—Various experiments have been made to determine coefficients of discharge for weirs of other than the standard sharp-crested form. Thus there have been used flat crests of various thickness, rounded crests, and various plane, curved and compound surfaces up-stream and down-stream from the crest. The variable elements are so numerous that no brief summary of results can be given.† Accurate estimates of discharge cannot be based upon such data unless the design of the original experimental weir is closely duplicated.

Waste weirs.—A waste weir is a notch left at the crest of a dam for the purpose of discharging flood-water without injury to the dam. Since the required maximum rate of discharge will be known only roughly, an approximate formula suffices for estimating the requisite length and depth of a waste weir. For a crest of rectangular section exceeding 2 ft. in thickness it may be assumed that the discharge will be about 80% as great as for a standard weir; while for rounded crest the discharge may be nearly or quite as great as for the standard weir, or may even slightly exceed it.

153. Triangular Weir.—Although the rectangular form of weir is generally the most convenient, a triangular notch may be used when the quantity to be measured is not too great. Theoretically the triangular form has the advantage that the shape of the water section does not change with the head, so that the coefficient of discharge would be expected to be more nearly constant with varying head than in the case of the rect-

* If F = cross-section of channel at place where H is measured, the value of G for a uniform channel giving the same mean velocity of approach may be found from the equation $b(G+H)=F$.

† See U.S. Geol. Survey, Water Supply and Irrigation Paper No. 200.

angular form. This is confirmed by experiment.

The formula for the rate of discharge over a triangular notch, as deduced in Art. 51, is

$$q=c\frac{4}{15}\sqrt{2g}\,bH^{\frac{3}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

If the angle at the vertex is 90°, $b=2H$, and the formula becomes

$$q=c\frac{8}{15}\sqrt{2g}\,H^{\frac{5}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

If the edges are made sharp, as in the case of the standard rectangular weir, the value of c for heads under 0.8 ft. may be taken as .592. With the foot as unit length the formula may be written

$$q=2.54H^{\frac{5}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

EXAMPLES.

1. Compute the rate of discharge of an orifice 6″ square when the head on the center is 18 ft. *Ans.* 5.12 cu. ft. per sec.

2. Estimate the diameter of a circular orifice to discharge .1 cu. ft. per sec., the head on the center being 5 ft. *Ans.* 0.345 ft.

3. A weir without end contractions is 4 ft. long, the bottom of the channel of approach being 3 ft. below the crest. Compute the rate of discharge when $H=0.75$ ft. (*a*) by formula of Francis and (*b*) by formula of Bazin. What percentage of error is caused by neglecting velocity of approach? *Ans.* (*a*) 8.73 cu. ft. per sec. (*b*) 8.96 cu. ft. per sec.

4. A contracted weir 9 ft. long is to discharge 60 cu. ft. per sec. Using formula of Francis, estimate the head on the crest, neglecting velocity of approach. *Ans.* $H=1.630$ ft.

5. Water enters a reservoir at the rate of 20 cu. ft. per sec. In the side is a rectangular notch 3.5 ft. long, with ends and crest so formed as to allow complete contraction. Estimate the head on the crest when a steady condition is attained. *Ans.* $H=1.52$ ft.

6. A rectangular orifice 1 ft. wide and 4 in. deep is formed in the vertical side of a reservoir into which water flows at the rate of 5 cu. ft. per sec. How high above the center of the orifice will the water surface rise? *Ans.* About 9.6 ft.

7. Compute the rate of discharge of a triangular notch having a vertex angle of 90° when the head is 0.46 ft. *Ans.* 0.365 cu. ft. per sec.

154. Submerged Weir.—A weir is said to be submerged if the surface of the stream below is higher than the crest of the weir. To deduce a formula for the rate of discharge in such a case, let H, h denote the elevations of the up-stream and down-stream surfaces respectively above the crest, and b the horizontal length of the weir (Fig. 75). The total discharge orifice $ABFE$ consists of two parts, $ABDC$, $CDFE$, of which the former may be treated as an ordinary weir and the latter as a submerged orifice. Omitting coefficients of discharge, the formula for the ordinary weir $ABDC$ is

$$q=\tfrac{2}{3}\sqrt{2g}\,b(H-h)^{\frac{3}{2}},$$

and that for the submerged orifice $CDFE$ is

$$q=bh\sqrt{2g(H-h)}.$$

Combining these, and introducing the coefficient of discharge c, the formula for the rate of discharge of the submerged weir becomes

$$q=c\tfrac{2}{3}\sqrt{2g}\,b(H-h)^{\frac{1}{2}}(H+\tfrac{1}{2}h). \quad . \quad . \quad . \quad . \quad (15)$$

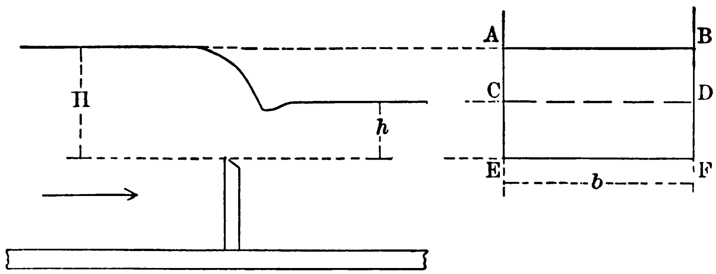


FIG. 75.

Equivalent ordinary weir.—If H' is the head on an unsubmerged weir whose rate of discharge is q, we have

$$q=c\tfrac{2}{3}\sqrt{2g}\,bH'^{\frac{3}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Assuming the coefficients of discharge for the two cases to be equal, the values of q given by (15) and (16) will be equal if

$$H'^3 = (H-h)(H+\tfrac{1}{2}h)^2 = H^3 - \tfrac{3}{4}Hh^2 - \tfrac{1}{4}h^3,$$

or

$$\left(\frac{H'}{H}\right)^3 = 1 - \frac{3}{4}\left(\frac{h}{H}\right)^2 - \frac{1}{4}\left(\frac{h}{H}\right)^3. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Writing n for H'/H, and assuming c to have the mean value found by Francis for ordinary weirs, the formula for the rate of discharge over a submerged weir without end contractions may be written

$$q = 3.33b(nH)^{\frac{3}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

in which n is to be computed from (17).

The relation between h/H and n as given by equation (17) is shown by the curve (B) in Fig. 76. Curve (A) shows the

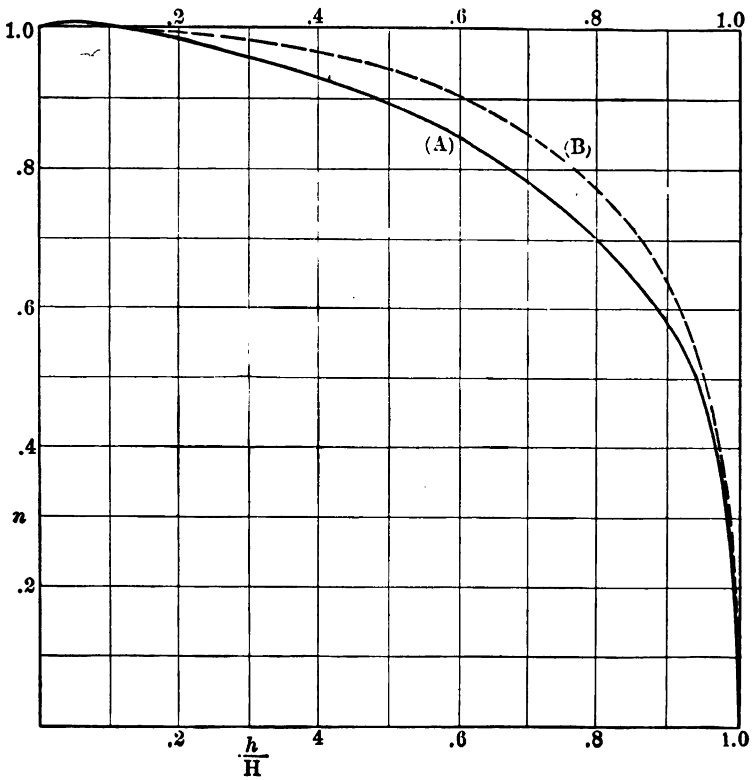


FIG. 76.

relation deduced by Herschel * from experiments made by Francis and by Ftely and Stearns. These results are also given in Table XII.

155. Submerged Dam.—If the channel of a stream is obstructed by a dam, the crest of which is lower than the original surface of the stream, the surface up-stream from the obstruction will be raised an amount depending upon the position of the crest, the length of the overfall, and the rate of discharge. An approximate estimate of the effect may be made by the formula above given for submerged weirs.

TABLE XII.

$\frac{h}{H}$	n Theoretical.	n Experimental.	$\frac{h}{H}$	n Theoretical.	n Experimental.	$\frac{h}{H}$	n Theoretical.	n Experimental.
0	1.000	1.000	.45	.959	.912	.84	.724	.656
.05	1.000	1.007	.50	.945	.892	.86	.697	.631
.10	.999	1.005	.55	.928	.871	.88	.666	.604
.15	.997	.996	.60	.908	.846	.90	.631	.574
.20	.995	.985	.65	.883	.818	.92	.589	.539
.25	.991	.972	.70	.853	.787	.94	.538	.498
.30	.986	.959	.75	.816	.750	.96	.473	.441
.35	.979	.944	.80	.770	.703	.98	.378	.352
.40	.970	.929	.82	.748	.681	1.00	.000	.000

As an example, suppose a stream discharging 600 cubic feet per second is obstructed by a dam with an overfall 40 feet long, the crest being 2 feet below the original water surface. The increase in depth above the dam may be estimated as follows:

With the notation of Art. 154, the value of H', the head on the equivalent weir, is found from the equation

$$600 = 3.33 \times 40 H'^{\frac{3}{2}},$$

or

$$H' = nH = 2.728 \text{ ft.}$$

Also, since $h = 2$ ft.,

$$\frac{nH}{h} = \frac{2.728}{2} = 1.364,$$

* Transactions American Society of Civil Engineers, Vol. XIV, p. 189.

or
$$n = 1.364\frac{h}{H}.$$

By trial substitutions in the table, using the experimental values of n, it is found that $h/H = .615$ very nearly, from which $H = 3.25$ ft.

If the curves in Fig. 76 be carefully drawn on cross-section paper, the above solution may be shortened. The equation $n = 1.364\frac{h}{H}$ represents a straight line whose intersection with curve (A) determines n and h/H at once.

As a second example, let it be required to determine where the crest of the dam should be placed in order to raise the surface 1 foot, the data being otherwise as above given. From the condition stated,

$$H - h = 1 \text{ ft.},$$

and as before

$$H' = nH = 2.728 \text{ ft.},$$

or
$$H = \frac{2.728}{n}.$$

From these equations,

$$1 - \frac{h}{H} = \frac{n}{2.728}.$$

By trial substitution in the table it is found that $n = .775$, from which $H = 3.52$ ft. and $h = 2.52$ ft.

This case may also be solved graphically, by finding the intersection of the straight line $1 - \frac{h}{H} = \frac{n}{2.728}$ with curve (A) in Fig. 76.

EXAMPLES.

1. A stream discharging 550 cu. ft. per sec. is obstructed by a dam of which the crest is 2.5 ft. below the original water surface, the length of the overfall being 50 ft. What will be the effect on the water surface above the dam? *Ans.* $H-h=0.63$ ft.

2. Where must the crest of a dam be placed in order that the water may be raised 2 ft., the length of the overfall being 30 ft. and the rate of discharge 500 cu. ft. per sec.? *Ans.* $h=1.10$ ft.

156. Venturi Meter.—The essential features of the Venturi meter have already been shown in Figs. 35 and 36, illustrating examples given at the end of Chapter V. The instrument consists merely of a converging tube with piezometers connected substantially as shown in Fig. 77, or a difference-gauge, as in Fig. 78.

Referring first to the arrangement in Fig. 77, let the usual notation be used for cross-sectional areas, velocities, and heights of piezometer columns, and let the equation of energy be written, A being the up-stream and B the down-stream section. Then,

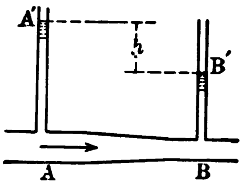


FIG 77.

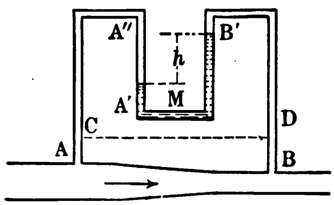


FIG. 78.

whether the axis of the pipe be horizontal or not, we have, as in Art. 125,

$$y_1+\frac{v_1^2}{2g}+k_1=y_2+\frac{v_2^2}{2g}+k_2+H',$$

H' being the head lost between A and B, and k_1, k_2 being positive quantities whose values depend upon the distribution of velocities throughout the cross-sections. Neglecting k_1, k_2 and H', and introducing the relation $F_1v_1=F_2v_2$, there results the equation

$$v_2 = c\sqrt{\frac{2g(y_1 - y_2)}{1 - \left(\frac{F_2}{F_1}\right)^2}},$$

in which c is a coefficient whose value would be 1 if the theory were perfect, and whose value as found by experiment usually ranges from 1 to 0.97. The value of c decreases with increasing velocity.

The factor $y_1 - y_2$ in the above formula is the difference in level between the tops of the two piezometer columns A' and B' (Fig. 77). If the pressure is too great for the use of open piezometers, a difference-gauge may be connected as in Fig. 78. If the specific gravity of the mercury or other fluid in the U tube is s, and if h is the difference in level of the two columns, as indicated in Fig. 78, the value of $y_1 - y_2$ in the formula for v_2 is $(s-1)h$. For mercury the value of s is very nearly 13.6.

EXAMPLES.

1. A Venturi meter whose two diameters were 48 inches and 16 inches was furnished with piezometers as in Fig. 77. The difference in level between the tops of the two columns was found to be 1.27 ft. in a certain case. Compute the rate of discharge, assuming $c = 0.985$.

Ans. 12.5 cu. ft. per sec.

2. A Venturi meter of the dimensions given in Ex. 1 was furnished with a mercury difference-gauge, as in Fig. 78. In a certain experiment the difference in elevation of the two mercury columns in the U tube was 0.625 ft. Estimate the rate of discharge, taking $c = 0.98$.

Ans. 31.0 cu. ft. per sec.

CHAPTER XIV.

DYNAMIC ACTION OF STREAMS.

157. Meaning of Dynamic Action.—The object of the present chapter is to investigate the forces which are exerted by a stream upon bodies which constrain its motion, and the reactions exerted by these bodies upon the stream.

Thus, if a free jet of water is deflected, forces are called into action between the particles of the jet and the body causing the deflection. The same is true in the case of a confined stream whose velocity varies in magnitude from section to section because the cross-section of the pipe changes, or in direction because the pipe bends. These are illustrations of dynamic action in cases of steady flow.

Another important case is that in which the rate of discharge of a confined stream, and therefore the velocity at every section, are suddenly changed, as by the closing of a valve.

In all cases, dynamic effects are to be determined by applying the fundamental laws of motion or the equations based upon them.

158. Principle of Momentum.—If a constant force P acts upon a body of mass m for a time Δt, thereby giving its velocity * the increment ΔV, we have

$$P=\frac{m\Delta V}{\Delta t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

* Throughout the remainder of this book it is necessary to distinguish between absolute and relative velocities of a stream. We therefore use V for the former and v for the latter. See Appendix B for a discussion of absolute and relative motion.

If the force is not constant, the equation gives its average value during the time Δt.

Thus if a body of 12 lbs. mass ($m=12/g=.373$ if force is to be expressed in pounds) has its velocity changed in .1 sec. from 18 ft. per second in a given direction to 23 ft. per second in the same direction, the average value of the force during that time is

$$P=\frac{.373(23-18)}{.1}=18.6 \text{ lbs.}$$

If $\Delta t=1$ sec., the equation is

$$P=m\Delta V. \quad \ldots \ldots \ldots \quad (2)$$

That is, the change of momentum produced by a force in one second is equal to the value of the force if constant, or to its average value for that second if variable.

159. Force Producing Velocity of Jet from Orifice.—Let W lbs. of water per second flow from an orifice in the side of a reservoir, V being the velocity of the jet. The velocity of each particle is then changed by the amount V during its passage from the reservoir to the smallest cross-section of the jet. The force causing this change is exerted directly by contiguous particles, but indirectly by the walls of the reservoir. The values of the forces exerted upon individual particles cannot be determined, but the sum of the average forces for all particles can be computed from the principle of momentum above stated.

In one second the mass of water flowing from the orifice is W/g, and the total momentum produced in one second by the action of the reservoir upon the water is WV/g. This is therefore the average value of the total force continually exerted by the reservoir upon the stream. This force is practically constant since the flow is steady, hence its average value is its actual value. If F is the cross-section of the jet, $W=wFV$, and the force is

$$P=\frac{wFV^2}{g}. \quad \ldots \ldots \ldots \quad (3)$$

160. Reaction of Jet upon Reservoir.—By the law of action and reaction, the particles of the jet exert upon the reservoir forces whose resultant is equal and opposite to the force just computed.

EXAMPLES.

1. Determine the reaction upon the reservoir due to a jet from a standard circular orifice 1″ in diameter under a head of 3′.

Ans. 1.21 lbs.

2. Show that if there were no loss of energy, the reactive force exerted by a jet from an orifice would be double the total static pressure upon an area equal to the cross-section of the jet, due to the head on the orifice.

161. Jet Striking a Fixed Surface Normally.—If a jet is intercepted by a plane surface perpendicular to the axis of the jet, the resultant action and reaction between the stream and the body will be directed normally to the surface. If V is the velocity of the jet, the increment of velocity for each particle, resolved in the direction of the normal, is $-V$, so that the resultant force exerted by the body upon the jet is

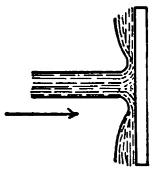


FIG. 79.

$$P=\frac{WV}{g}=\frac{wFV^2}{g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

in the direction opposite to the motion of the jet. The reaction of the jet upon the fixed body is equal and opposite to this.

162. Jet Striking a Moving Surface Normally.—Let a jet whose velocity is V strike normally against the surface of a body which is itself moving with velocity u in the same direction as the jet. The increment of velocity for each particle, resolved normally to the surface, has the magnitude $V-u$. If W' denotes the number of pounds of water striking the surface per second, the force exerted upon the moving body has the value

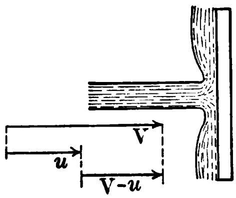


FIG. 80.

$$P'=\frac{W'}{g}(V-u).$$

Or, since
$$W' = wF(V-u),$$

$$P' = \frac{wF}{g}(V-u)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Work done upon moving body.—The work done in one second by the force P' is

$$P'u = \frac{wF}{g}u(V-u)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

163. Jet Water Wheel with Flat Vanes.—Fig. 81 represents a wheel provided with flat vanes against which a jet of water is directed. The results of Art. 162 would be strictly applicable if the vanes always received the jet normally. Although this condition cannot be realized, the above results may be used as an approximation, and equations (5) and (6) may be employed to compute the force exerted upon a vane, and the work done by this force per second. The total action of the jet upon the wheel is not, however, the same as its action upon one vane, since more than one vane will be receiving the jet at the same time.

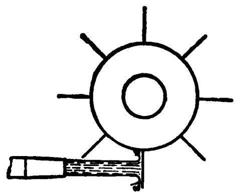


FIG. 81.

Replacing W' by W, the weight of water discharged by the jet per second, we get for the force exerted upon the wheel

$$P = \frac{W}{g}(V-u) = \frac{wF}{g}V(V-u), \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and for the work done upon the wheel per second

$$L = Pu = \frac{wF}{g}Vu(V-u). \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Maximum value of work done upon wheel.—If u varies, the value of the work varies, having its maximum when $u = V/2$, which gives

$$\text{Maximum } L = \frac{wFV^3}{4g} = \frac{WV^2}{4g}. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Since the kinetic energy of W lbs. of water in the jet is $WV^2/2g$, it is seen that not more than half of this can be utilized by a water wheel with flat vanes.

164. Force Causing Deflection of a Jet.—If the velocity of a particle changes in direction, the average value of the force acting upon it during a time Δt is still given by the formula

$$P=\frac{m\Delta V}{\Delta t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

but ΔV is now the *vector* increment * of velocity.

In Fig. 82 is represented a jet of water which is deflected by a curved vane MN, the velocity of every particle being changed from V_1 (represented by the vector OA) to V_2 (represented by the vector OB). The increment of velocity is then represented by the vector AB. If the discharge of the jet is W lbs. per second, the change of momentum produced in one second by the action of the deflecting surface is

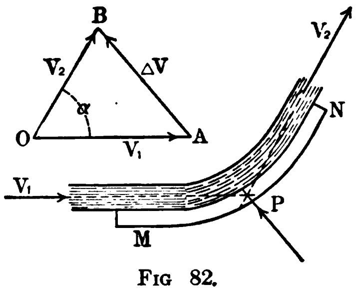


FIG 82.

$$\frac{W}{g}\Delta V,$$

which is therefore the value of the resultant force constantly exerted upon the jet by the deflecting body. If V_1, V_2 and the angle of the deflection α are known, the magnitude and direction of ΔV can be computed by solving the triangle OAB. The line of action of the resultant force passes through the intersection of the two lines which coincide with the axis of the jet before and after the deflection, as indicated in Fig. 82.

If the magnitude of the velocity is not changed, so that $V_2=V_1=V$,

$$\Delta V=2V \sin\frac{\alpha}{2},$$

* Theoretical Mechanics, Art. 253.

and $$P=\frac{W}{g}\Delta V=\frac{2WV}{g}\sin\frac{\alpha}{2}=\frac{2wFV^2}{g}\sin\frac{\alpha}{2}, \quad . \quad . \quad (11)$$

α being the angle of deflection, i.e., the angle between V_1 and V_2.

In any case, the reaction of the jet upon the body which deflects it is equal and opposite to the force exerted upon the jet.

EXAMPLES.

1. A jet of 2 sq. in. cross-section, having a velocity of 40 ft. per sec., strikes a curved surface which deflects it 25° and changes its velocity to 35 ft. per sec. Determine the magnitude and direction of the force exerted by the jet upon the deflecting body.

Ans. A force of 19.3 lbs. making angle of 60° 46′ with jet.

2. If the jet in Ex. 1 is deflected 25° without diminution of velocity, determine the force exerted upon the deflecting body.

Ans. 18.7 lbs. making angle of 77° 30′ with jet.

165. Case in which the Velocity of the Stream is Reversed. —If the deflection of the stream amounts to 180°, as in Fig. 83, the increment of velocity is V_1+V_2, and the force exerted by the deflecting body is

$$P=\frac{W}{g}(V_1+V_2)=\frac{wF_1}{g}V_1(V_1+V_2), \quad . \quad . \quad . \quad (12)$$

if F_1 is the cross-section of the stream where its velocity is V_1.

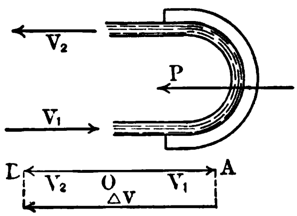


FIG. 83.

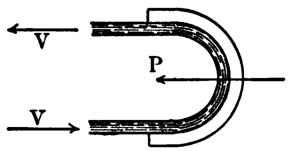


FIG. 84.

If the jet could be reversed without diminution of its velocity (Fig. 84), so that $V_2=V_1=V$, the force would have the value

$$P=\frac{2WV}{g}=\frac{2wFV^2}{g}. \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The line of action of P is found from the law of composition of parallel forces. The momentum WV_2/g is the resultant of the

momentum WV_1/g and that due to the force P acting for one second. The distance between the lines lying in the axis of the jet before and after deflection is therefore divided by the line of action of P in the inverse ratio of V_1 and V_2.

166. Reversal of Jet by Moving Vane.—In the preceding case suppose the curved vane which deflects the jet to be moving with velocity u in the same direction as the impinging jet. If V_1 and V_2 are the values of the *absolute* velocity of the stream just before striking the vane and just after leaving it, and if W' denotes the weight of water striking the vane per second, the force exerted by the jet upon the vane is

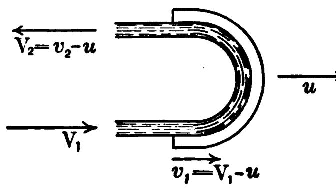


FIG. 85.

$$P' = \frac{W'}{g}(V_1 + V_2).$$

The values of V_2 and of W' will depend upon u.

Let v denote the velocity of a particle *relative to the vane* at a point where its absolute * velocity is V; v_1 being the value of v at the point where the stream is about to strike the vane, and v_2 its value at the point where the stream leaves the vane. Then

$$v_1 = V_1 - u;$$
$$V_2 = v_2 - u.$$

If there were no loss of energy in the flow over the vane, v_2 would equal v_1. Assuming this to be true,

$$V_2 = V_1 - 2u,$$
$$V_1 + V_2 = 2(V_1 - u).$$

Substituting in the above value of P', and noticing that

$$W' = wF_1(V_1 - u),$$

we have
$$P' = \frac{2wF_1}{g}(V_1 - u)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

* See Appendix B.

Work done on moving vane.—The work done per second by the force P' is

$$P'u=\frac{2wF_1}{g}u(V_1-u)^2. \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Comparing with Art. 162, it is seen that these values of the force and the work are double those found in the case of a flat vane.

167. Jet Water Wheel with Curved Vanes. — If the flat vanes in Fig. 81 were replaced by curved vanes so arranged as to receive and discharge the jet substantially as in Fig. 85, the above reasoning would hold approximately, equations (14) and (15) representing the force exerted upon a single vane and the work done by that force per second. Since, however, more than one vane would be receiving water at the same time, so that the wheel would be acted upon by all the water discharged by the jet, W' must be replaced by W to get the total action upon the wheel, in which

$$W=wF_1V_1.$$

That is, the total force exerted upon the wheel would be

$$P=\frac{W}{g}(V_1+V_2)=\frac{wF_1}{g}V_1(V_1+V_2)=\frac{2wF_1}{g}V_1(V_1-u); \quad (16)$$

and the work done upon the wheel per second would be

$$L=Pu=\frac{2wF_1}{g}V_1(V_1-u)u. \quad . \quad . \quad . \quad . \quad (17)$$

These values are double those obtained in Art. 163 for a wheel with flat vanes.

The maximum value of the work, given by $u=V_1/2$, is

$$\text{Maximum } L=\frac{WV_1^2}{2g}, \quad . \quad . \quad . \quad . \quad . \quad (18)$$

which is equal to the whole kinetic energy of the jet.

This form of water wheel is discussed in Chapter XVIII.

168. Principle of Angular Momentum.—From the principle stated in Art. 158 another of equal importance may be deduced.

Since the total increment of momentum of a particle per second is equal to the average value of the force acting upon it, it follows by taking moments about any axis that the total increment of the moment of momentum (or "angular momentum") per second is equal to the average value of the moment of the force.

This principle is of use in the following discussion.

169. Action of Stream upon Rotating Vane in General Case.—All the special cases above considered are covered by the following general discussion.

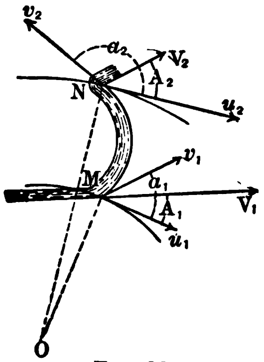


FIG. 86.

Let MN (Fig. 86) represent a curved vane which rotates uniformly, with angular velocity ω, about an axis at O perpendicular to the plane of the figure. Let the vane receive a stream at M and discharge it at N. The point M of the vane moves at every instant at right angles to OM with velocity u_1, and the point N moves perpendicularly to ON with velocity u_2.

Let V_1, v_1 = absolute and relative * velocities of a particle of water at M, just before striking the vane;

V_2, v_2 = absolute and relative velocities of a particle at N, just leaving the vane;

A_1, a_1 = angles between V_1, v_1 respectively and u_1,

A_2, a_2 = " " V_2, v_2 " " u_2.

In order to compute the action of the jet upon the vane, we may apply the principle of angular momentum, taking the axis of rotation O as axis of moments. Let $OM = r_1$, $ON = r_2$.

The angular momentum of a particle of water of mass m just before striking the vane is

$$mV_1r_1 \cos A_1,$$

* By "relative velocity" will always be meant "velocity relative to the rotating body," as described in Appendix B.

and its angular momentum just as it leaves the vane is

$$mV_2r_2 \cos A_2,$$

so that the action of the vane changes the angular momentum of each particle by the amount

$$m(V_2r_2 \cos A_2 - V_1r_1 \cos A_1).$$

The total change of angular momentum due to the action of the vane for one second is

$$\frac{W'}{g}(V_2r_2 \cos A_2 - V_1r_1 \cos A_1),$$

if W' is the weight of water striking the vane per second. This is therefore also the average value of the *total moment of the forces* exerted by the vane upon the water.

If there is a succession of similar vanes forming a water wheel, the total action upon the wheel is found by replacing W' by W, the total weight of water used by the wheel per second. If G denotes the total moment of the forces exerted by the water upon the wheel, we have

$$-G = \frac{W}{g}(V_2r_2 \cos A_2 - V_1r_1 \cos A_1),$$

or

$$G = \frac{W}{g}(V_1r_1 \cos A_1 - V_2r_2 \cos A_2). \quad . \quad . \quad . \quad (19)$$

Applications of this result will be given in the following chapters.

170. Force Exerted upon Pipe by Confined Stream in Steady Flow.—Let AB (Fig. 87) represent a portion of a pipe carrying a steady stream, F_1, V_1 being the cross-section and velocity at A and F_2, V_2 the like quantities at B. Let the rate of discharge be W pounds per second.

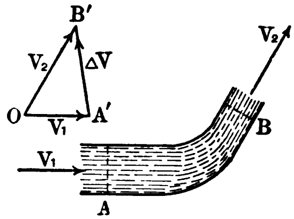


FIG. 87.

In passing from A to B a particle of mass m receives an increment of momentum $m\Delta V$, in which $\Delta V = A'B'$ (Fig. 87), V_1 and V_2 being represented by the vectors OA', OB'. Hence the average

force exerted upon the particle is

$$\frac{m\Delta V}{\Delta t},$$

if Δt is the time required for this change of momentum. (Art. 158.) The resultant of such average forces for all particles will be constant, and may be computed by considering the body of water which at any instant occupies the volume AB, and following its motion for the time dt. During this time equal weights of water Wdt pass the sections A and B, the momentum of the former being $\frac{W}{g}V_1\,dt$, and that of the latter $\frac{W}{g}V_2\,dt$. The flow being steady, the vector difference between these values, or $\frac{W\Delta V}{g}dt$, is the total change of momentum of the body under consideration during the time dt. The resultant force acting upon the body is therefore

$$P=\frac{W\Delta V}{g}=\frac{W}{g}\text{ (vector } A'B'). \quad . \quad . \quad . \quad . \quad (20)$$

The forces which make up this resultant are the weight of the water, the pressures exerted by the adjacent water upon the cross-sections A and B, and the forces exerted by the portion of the pipe between A and B. The resultant of these latter forces can therefore be determined if the pressures in the stream at A and B are known. The equal and opposite reaction to this force is the force exerted by the stream upon the portion AB of the pipe.

EXAMPLES.

1. Let the axis of the pipe in Fig. 88 be horizontal, the diameter at A being 1 ft. and at B .5 ft. Assume the rate of discharge to be 2 cu. ft. per second in the direction AB, and the pressure head at the center of the section A to be 12 ft. Compute the resultant force exerted upon the pipe by this portion of the stream, on the assumption of no loss of head.

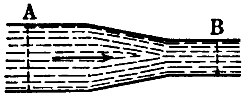


FIG. 88.

From the given data $V_1=2.55$, $V_2=10.2$, $V_1^2/2g=.101$, $V_2^2/2g=1.62$. Assuming no loss of head,

$$\frac{p_1}{w}+\frac{V_1^2}{2g}=\frac{p_2}{w}+\frac{V_2^2}{2g},$$

from which $p_2/w=10.5$ ft.

The increment of momentum per second for the body AB is

$$\frac{W}{g}(V_2 - V_1) = \frac{62.5 \times 2 \times (10.20 - 2.55)}{32.2} = 29.7;$$

hence the resultant of all forces acting upon this body is 29.7 lbs. in the direction AB. This is made up of the pressures on the two cross-sections A and B and the forces exerted by the pipe. Calling the resultant of the latter $-P$ (so that P is the force exerted by the stream upon the pipe), we have

$$29.7 = p_1F_1 - p_2F_2 - P,$$

or

$$P = p_1F_1 - p_2F_2 - 29.7 = 589 - 129 - 29.7 = 430 \text{ lbs.}$$

2. Generalizing the preceding example, show that the resultant force exerted by the stream upon the pipe AB, neglecting loss of head, is

$$P = w(F_1 - F_2)\left[\frac{p_1}{w} - \left(\frac{F_1}{F_2} - 1\right)\frac{V_1^2}{2g}\right]$$

$$= w(F_1 - F_2)\left[\frac{p_2}{w} - \left(\frac{F_2}{F_1} - 1\right)\frac{V_2^2}{2g}.\right]$$

3. Solve with data as in Ex. 1, except that the flow is in the opposite direction.

4. A stream is discharged into the atmosphere through a nozzle which reduces the diameter from 2″ to .5″. If the pressure head at the nozzle entrance is 100′ above atmospheric, compute the resultant force exerted by the stream upon the nozzle, assuming no loss of head.

Ans. 122 lbs.

5. A straight horizontal pipe 1′ in diameter is discharging 1 cu. ft. per sec. Assuming that the loss of head is given by Dárcy's formula, determine the resultant force exerted by the stream upon the pipe in a length of 100′.

Ans. 2.67 lbs.

171. Theory of Pitot's Tube.—The measurement of the velocity at any point in a stream by means of Pitot's tube has been referred to in Chapter XIII. Fig. 89 represents such a tube placed with the lower end facing the current, the main part of the tube being vertical and the upper end projecting above the water surface. Let V denote the velocity of the current passing the lower end of the tube, and h the height to which the water in the tube rises above the surface of the stream. Then h measures the excess of the

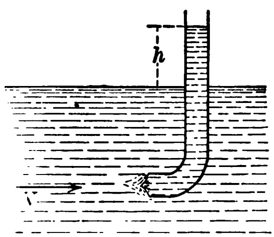


FIG. 89.

pressure at any point within the tube above the pressure at the same level in the stream outside the tube. If F is the cross-sectional area of the tube at the end facing the stream, the dynamic action of the stream upon this area must amount to a force

$$whF$$

in order to maintain the column within the tube in equilibrium. The effect of the tube is to deflect a certain part of the current, and the total change of momentum thus produced in one second is equal in magnitude to the force exerted upon the tube and its contained water by the water which is deflected. (Art. 158.) If W is the weight of water deflected per second and ΔV its average increment of velocity, the force would be

$$\frac{W\Delta V}{g}.$$

It seems reasonable to assume that ΔV is proportional to V, and that the part of W whose deflection is due to impingement against the water at rest in the tube is proportional to wFV, so that the force exerted by the current upon the area F is proportional to

$$\frac{wFV^2}{g}.$$

Comparing with the value given above,

$$whF \text{ varies as } \frac{wFV^2}{g},$$

or

$$h = k\frac{V^2}{2g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

in which k is a coefficient which would be constant if the theory were exactly true. For the instrument constructed with two tubes, as described in Art. 145, equation (21) is applied with h denoting the difference of elevation of the two columns.

The above theory indicates that k is independent of F, but depends upon the relation between ΔV and V; and this would appear to depend upon the external form of the tube rather than upon the form of the bore. The actual value of the coefficient must be determined experimentally for each instrument. Darcy found $k = 1.41$ for an instrument in which the tube facing the current was cylindrical, while Bazin gave $k = 1$ for one in which the tube was conically convergent; in each the opening of the second tube was directed at right angles to the current. Other authors have given values ranging from 1 to 2.

172. Ram Pressure.—The sudden stoppage of the flow in a pipe may cause pressures much greater than those due to the static head. Pressure due to this cause is called ram pressure, or "water-ram."

In Fig. 91 let the flow in the pipe be steady, the velocity being V, and suppose a valve at C to be suddenly closed, thus checking and finally stopping the flow. At any point upstream from the valve the pressure is increased by an amount which is less as the distance from the valve is greater. The amount of increase at any point depends upon the rate of decrease of the velocity. This effect is strictly analogous to that due to a moving body which is brought to rest by striking a fixed obstacle; it exerts upon the obstacle a force proportional directly to the mass of the moving body and to the rate at which its velocity is decreased. The value of the ram pressure at any point, in the ideal case of an incompressible fluid and an unyielding pipe, may be expressed in the following manner.

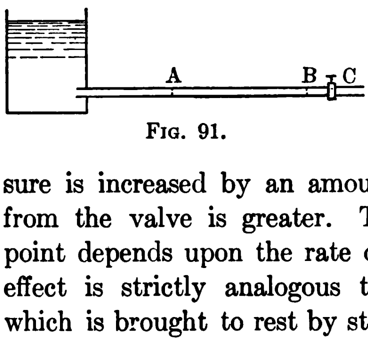


FIG. 91.

Let AB be any straight portion of the pipe of length l, and let p_1, p_2 denote the values of the pressure at A and B. If m is the total mass of the cylinder of water AB, the resultant of all forces acting upon it at any instant is (by the fundamental equation of motion)

$$P = m\frac{dV}{dt}. \quad \ldots \ldots \quad (22)$$

While the flow is steady this resultant is zero, since $dV/dt=0$, but while the velocity is changing P is not zero. In the present problem the forces called into action by the sudden closing of the valve are great in comparison with those acting when the flow is steady, and the latter may be neglected as in all cases in which impulsive forces are considered.* The only forces to consider are, therefore, the pressures caused by the sudden stoppage of the flow. The resultant of these pressures acting upon the body AB amounts to a force

$$P=(p_1-p_2)F$$

in the direction AB, F being the cross-section of the pipe. But also, as above,

$$P=m\frac{dV}{dt}=\frac{wFl}{g}\cdot\frac{dV}{dt}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

therefore
$$p_2-p_1=-\frac{wl}{g}\cdot\frac{dV}{dt},$$

or
$$\frac{p_2}{w}-\frac{p_1}{w}=-\frac{l}{g}\cdot\frac{dV}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Since dV/dt is negative (because V decreases), the second member of this equation is really positive, and p_2 is greater than p_1.

This formula shows that the ram pressure varies along the pipe directly as the length, and is independent of the diameter for a given value of dV/dt. The result is seen to hold even if the pipe is curved, since the length may be subdivided into elements in applying the above reasoning.

If the pipe leads from a reservoir, as in Fig. 91, the ram pressure will be zero at the intake end, so that for a point distant l from the intake end we may write (pressure being expressed in terms of equivalent water column)

$$\text{Ram pressure}=-\frac{l}{g}\cdot\frac{dV}{dt}. \quad . \quad . \quad . \quad . \quad . \quad (25)$$

* Theoretical Mechanics, Art. 321.

No practical use can be made of this formula unless dV/dt can be estimated. Experiment indicates that the closing of a valve produces no ram pressure of importance except during the last part of the closing; that is, the rate of change of the velocity does not become great until near the very end of the process. Dangerous ram pressures may therefore be avoided by using valves so constructed that the last part of the closing must be slow.

The above results are doubtless modified in an important degree by the elastic yielding of the pipe and of the water. It is doubtful whether a theoretical discussion taking account of these factors can be given which will accord closely with practical conditions.

EXAMPLE.

1. If the velocity of flow in a pipe is changed in .25 sec. from 2 ft. per sec. to 0 by the closing of a valve 1000 ft. from the reservoir, estimate the average value of the ram pressure close to the valve.

Ans. $p/w = 248$ ft.

CHAPTER XV.

THEORY OF STEADY FLOW THROUGH ROTATING WHEEL.

173. Statement of Problem. — Wheels designed for the utilization of the energy of streams of water are of various forms, and may be divided into several classes possessing some-

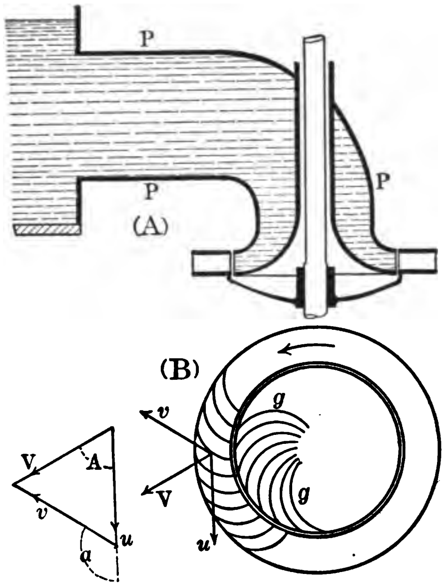


FIG. 92.

what distinct characteristics. The same basal theory, however, applies to practically all the important types. This theory it is the object of the present chapter to present; and for illustration reference will be made to the arrangement shown in Fig. 92.

The figure shows an elevation and vertical section (*A*) and

a plan and horizontal section (*B*). The "penstock" or supply-pipe *P* leads from the head race or supply reservoir and terminates in the guide passages *g g*. These are so formed as gradually to deflect the water outward and forward (i.e., in the direction of the rotation). A particle of water when leaving a guide passage is moving in a horizontal plane and in a direction determined by that of a guide vane. Within the wheel the path of a particle relative to the wheel is determined by the form of the wheel vanes. Arriving at the outlet of the wheel, the *relative* velocity of a particle has a direction determined by that of a wheel vane, while its *absolute* velocity depends upon this relative velocity and also upon the rotation of the wheel.

The problem whose solution is the basis of turbine theory is the following: Given all dimensions and the rotation speed of the wheel, required to determine the rate of flow through the wheel.

If the wheel were at rest, this would be a problem in ordinary hydraulics, to be solved by the methods illustrated in Chapter VI. It is to be shown how the rotation of the wheel affects the solution. The explanation involves the following points:

(1) The relation between the absolute and relative velocities of a particle; (2) the meaning of the general equation of energy in such a case as this; (3) the computation of the energy given up by the stream to the wheel; (4) the deduction from these results of a special form of the energy-equation involving relative instead of absolute velocities.

In the present chapter this problem will not be completely solved, since the details of the solution must be different for different types of wheel. In all cases, however, the solution involves certain general steps which are here outlined.

174. Notation.—The following notation will be employed throughout the entire discussion of turbines and water wheels:

r = distance of a particle from the axis of rotation;

z = its height above a horizontal datum plane;

v = its velocity relative to the wheel;

V = its velocity relative to the earth;

[Briefly, v and V will usually be called simply relative velocity and absolute velocity.]

ω = angular velocity of wheel (radians per second);

$u = r\omega$ = linear velocity of point of wheel momentarily coinciding with a particle of water under consideration;

s = resolved component of V in direction of u (called briefly tangential component of V);

a = angle between v and u;

A = angle between V and u;

q = volume of water discharged per second;

W = weight of water discharged per second;

w = weight of unit volume of water;

F = cross-section of stream at any point in the stationary part of the passages;

f = cross-section of stream at any point within the wheel;

L = energy imparted by water to rotating wheel per second.

The values of any of the foregoing quantities for different sections of the stream will be distinguished by suffixes. Thus suffix ($_1$) will refer to the stream leaving the guide passages, just before its motion is influenced by the wheel, and suffix ($_2$) to the stream leaving the wheel.

In many cases the stream is divided into several parts in passing through the wheel, also as it approaches the wheel through the guide passages. By F and f will be meant the sum of the cross-sections of all the partial streams taken at corresponding points in the different passages. Thus

F_1 = total cross-section of streams leaving guide passages, measured everywhere normally to the direction of V_1;

f_2 = total cross-section of streams leaving wheel passages, measured everywhere normally to the direction of v_2.

The equation of continuity (Art. 36) holds throughout the entire series of passages; thus

$$q = F_1 V_1 = f_2 v_2,$$

$$W = wq = wF_1 V_1 = wf_2 v_2.$$

In considering the relation between absolute and relative velocities it is often convenient to use the language of vector addition. For this purpose the vector value of a velocity will be represented by the use of brackets: $[u]$, $[v]$, $[V]$ = vector values of velocities whose magnitudes are u, v, V.

175. Relation between Absolute and Relative Velocities.*— The three velocities u, v, V are related in the manner shown by the vector triangle in Fig. 93. This relation is expressed by the statement that V is the vector sum of u and v, or

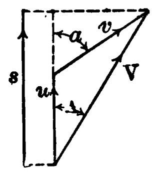


FIG. 93.

$$[V]=[u]+[v]. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Algebraically, any of the equations given by elementary trigonometry may be used for computing one of these vectors when the other two are known. The following are especially useful. Resolving along and perpendicularly to u (Fig. 93),

$$\left.\begin{array}{l} V \cos A = u + v \cos a, \\ V \sin A = v \sin a. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The tangential component of V is evidently

$$s = V \cos A = u + v \cos a. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

These relations being true for any particle, are true when suffix $(_1)$ or suffix $(_2)$ is attached to each symbol.

176. Application of General Equation of Energy. — The general equation of energy

$$H_1 - H_2 = H'$$

may be applied to any case of steady flow. In this equation H' means the quantity of energy lost by the water, between the sections $(_1)$ and $(_2)$, per pound of water discharged. (Art. 63.) In most of the applications hitherto considered this loss has been wholly due to dissipation of energy into heat. If the water gives up energy in any other way, the equation still holds,

* See Appendix B.

provided H' be made to include such loss as well as that due to dissipation into heat.

Thus, in the case shown in Fig. 92, let the section ($_1$) be taken where the water is leaving the guide passages, just before its motion is influenced by the rotating wheel, and the section ($_2$) at the point of outflow from the wheel. Then H' is made up of two parts, corresponding respectively to the energy lost by dissipation within the wheel and that given up to the wheel as mechanical energy. Calling these parts h' and h'' respectively, we have

$$H_1=\frac{V_1^2}{2g}+\frac{p_1}{w}+z_1,$$

$$H_2=\frac{V_2^2}{2g}+\frac{p_2}{w}+z_2,$$

$$H'=h'+h'',$$

and the equation may be written

$$\left(\frac{V_1^2}{2g}+\frac{p_1}{w}+z_1\right)-\left(\frac{V_2^2}{2g}+\frac{p_2}{w}+z_2\right)=h'+h''. \quad . \quad . \quad (4)$$

The equation applies also if the section ($_1$) be taken elsewhere, for example at the surface of the head race, proper values being given to V_1, p_1 and z_1, and it being understood that h' includes energy dissipated throughout the entire series of passages between ($_1$) and ($_2$), while the meaning of h'' is unchanged.*

177. Computation of Energy Given Up to Wheel. — The value of the energy given up to the wheel by the water may be expressed by a simple formula which will now be deduced.

The flow being steady and the rotation of the wheel uniform, the water exerts upon the wheel forces whose turning moment about the axis of rotation is constant. If G denotes this moment, the work done by the forces upon the wheel † while the latter turns through an angle θ (radians) is $G\theta$.

* See Art. 94. † Theoretical Mechanics, Art. 508.

The value of G may be determined from the principle of angular momentum, as in Art. 169. The value there found may be written (noticing that $V_1 \cos A_1 = s_1$ and $V_2 \cos A_2 = s_2$)

$$G = \frac{W}{g}(r_1 s_1 - r_2 s_2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The angle turned through in one second being ω, the work done on the wheel per second is $G\omega$; or, denoting by L the energy imparted to the wheel in one second, and noticing that $r_1\omega = u_1$, $r_2\omega = u_2$,

$$L = \frac{W}{g}(r_1 s_1 - r_2 s_2)\,\omega = \frac{W}{g}(u_1 s_1 - u_2 s_2). \quad . \quad . \quad . \quad (6)$$

178. New Form of General Equation of Energy.—From the meaning of h'' (Art. 176) it is evident that

$$L = Wh'',$$

so that

$$h'' = \frac{1}{g}(u_1 s_1 - u_2 s_2),$$

and equation (4) may be written

$$\left(\frac{V_1^2}{2g} + \frac{p_1}{w} + z_1\right) - \left(\frac{V_2^2}{2g} + \frac{p_2}{w} + z_2\right) = h' + \frac{1}{g}(u_1 s_1 - u_2 s_2). \quad (7)$$

This may be simplified by introducing relative instead of absolute velocities. Thus, from the vector relation between V, v and u (Fig. 93),

$$V_1^2 = v_1^2 + u_1^2 + 2u_1 v_1 \cos a_1,$$

$$V_2^2 = v_2^2 + u_2^2 + 2u_2 v_2 \cos a_2,$$

$$s_1 = u_1 + v_1 \cos a_1,$$

$$s_2 = u_2 + v_2 \cos a_2.$$

The substitution of these values in (7) reduces it to the form

$$\left(\frac{v_1^2-u_1^2}{2g}+\frac{p_1}{w}+z_1\right)-\left(\frac{v_2^2-u_2^2}{2g}+\frac{p_2}{w}+z_2\right)=h'. \quad . \quad (8)$$

This may be called the equation of energy for the relative motion.

It may be noticed that this equation includes as a special case the equation of energy in its ordinary form for flow through stationary passages, reducing to it when $\omega=0$; this makes u_1 and u_2 zero, and v_1 and v_2 become absolute velocities.

The meaning of h' should be kept clearly in mind. It is the energy lost by dissipation within the wheel; per pound of water discharged.

179. Summary of Principles.—The principles to be employed in the theory of turbines are embodied in the general formulas already deduced, which may be summarized here for convenience of reference.

The equation of continuity:

$$q=FV=fv=\text{constant}; \quad . \quad . \quad . \quad . \quad . \quad (\text{I})$$

and in particular

$$F_1V_1=f_2v_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{I}')$$

The relation between absolute and relative velocities of a particle, expressed by the vector equation

$$[V]=[u]+[v], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{II})$$

and by the algebraic equations

$$\left.\begin{array}{l} V\cos A=u+v\cos a, \\ V\sin A=v\sin a, \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{II}')$$

the two algebraic equations (II′) being equivalent to the single vector equation (II).

The equation of energy for steady flow in the ordinary form:

$$\left(\frac{V_1^2}{2g}+\frac{p_1}{w}+z_1\right)-\left(\frac{V_2^2}{2g}+\frac{p_2}{w}+z_2\right)=H'=h'+h''. \quad . \quad \text{(III)}$$

The formula for energy imparted to the wheel:

$$L=\frac{W}{g}(u_1s_1-u_2s_2). \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IV)}$$

The equation of energy for the relative motion:

$$\left(\frac{v_1^2-u_1^2}{2g}+\frac{p_1}{w}+z_1\right)-\left(\frac{v_2^2-u_2^2}{2g}+\frac{p_2}{w}+z_2\right)=h'. \quad . \quad \text{(V)}$$

180. Application of General Theory.—The application of the foregoing principles and formulas to particular cases involves special data pertaining to each case. The two cases of greatest practical importance are illustrated in the following examples, and the complete theory is treated in the succeeding chapters relating to the different types of turbines and water wheels.

The following examples refer to the arrangement shown in Fig. 92.

EXAMPLES.

1. A particle of water leaves the guide passage with a velocity of 22 ft. per sec. directed at angle 18° with the velocity u_1. The distance from the axis of rotation is 3.5 ft., and the wheel rotates at the rate of 90 R.P.M. Determine the magnitude and direction of the velocity of the particle relative to the wheel.

Ans. $v_1=13.85$ ft. per sec.; $a_1=150° \ 36'$.

2. In order that the particle (Ex. 1) shall not be suddenly deflected as it enters the wheel, what should be the direction of the wheel vane at the point where the water enters?

3. Using the data of Ex. 1, and assuming that the particle moves in a horizontal plane, that no frictional loss of energy occurs within the wheel, and that the pressure is constant throughout the wheel and at guide outlets, what will be the magnitude of the relative velocity of a particle when 4 ft. from the axis? What will determine the direction of this relative velocity?

Ans. $v=22.91$ ft. per sec.

4. Suppose the radius of the wheel at outlet is 4.5 ft., and $a_2 = 162°$. Making the same assumptions as in Ex. 3, determine the magnitude and direction of the relative velocity and of the absolute velocity of a particle leaving the wheel.

Ans. $v_2 = 30.05$ ft. per sec.; $V_2 = 16.66$ ft. per sec.; $A_2 = 33° 52'$.

5. With data as in Ex. 4, (*a*) how much energy does the wheel receive from the water for each pound of discharge? (*b*) If $F_1 = 6$ sq. ft., compute the H.P. imparted to the wheel. (*c*) The energy utilized is equivalent to what fall? *Ans.* (*a*) 3.21 foot-pounds.

[In the following examples, instead of supposing the pressure to be uniform throughout the wheel, assume that the wheel passage is completely filled at every cross-section, and that $F_1/f_2 = 1.5$, all other data being as in preceding examples. Answer the following questions.]

6. What further data are required for the determination of the relative and absolute velocities of a particle 4 ft. from the axis?

7. Determine the magnitude and direction of the absolute velocity and of the relative velocity of a particle as it leaves the wheel.

Ans. $v_2 = 33$ ft. per sec.; $V_2 = 15.03$ ft. per sec.; $A_2 = 42° 43'$.

8. If the discharge from the wheel takes place into the atmosphere, what pressure exists at guide outlets? *Ans.* $p_1/w - p_2/w = 2.9$ ft.

9. If $F_1 = 6$ sq. ft., compute H.P. imparted to wheel. What is the value of the "utilized head"?

Ans. H.P. $= 103.5$; utilized head $= 6.90$ ft.

CHAPTER XVI.

TYPES OF TURBINES AND WATER WHEELS.

181. Definition of Turbine.—There is no general agreement upon an exact definition of a turbine. By Bodmer * a turbine is defined as "a water wheel in which a motion of the water relatively to the buckets is essential to its action." Such a definition includes practically all modern wheels for the utilization of water power. The same author uses the term water wheel to designate the old-fashioned overshot, undershot, and breast wheels, which were usually of large diameter relatively to the fall utilized, and which were actuated either by the weight of the water directly or by the impact of a stream against flat vanes.

These definitions are not in conformity with present-day usage in America. This usage can be best explained after the different types of wheels have been described.† The word turbine will, however, generally be used with the broad meaning above given.

182. Classification of Turbines According to Direction of Flow.—The direction of flow of the water through a turbine may be either *radial*, *axial*, or *mixed*.

Radial flow means that the path of a particle within the wheel lies in a plane perpendicular to the axis of rotation. The direction of flow may be either outward (Fig. 92) or inward (Fig. 105).

Axial flow means that the distance of a particle from the

* Hydraulic Motors, p. 24. † See Art. 191.

axis of rotation remains constant during its passage through the wheel (Fig. 104).

Mixed flow is a combination of radial and axial flow. It is usually inward and axial, as in Fig. 106.

183. Classification into Impulse Wheels and Reaction Wheels.—If the total cross-section of the streams entering the wheel is so small that the wheel passages are not filled, and if air enters freely so that the entire stream within each wheel passage is under atmospheric pressure, the turbine is called an *impulse wheel.*

If the wheel passages are completely filled by the streams flowing through them, the turbine becomes a *reaction wheel.*

This is the most fundamental distinction between different wheels of modern type.

184. Complete and Partial Admission.—Water may be admitted to all the wheel passages at once, or to a limited number of them. Reaction wheels necessarily have complete admission, while impulse wheels may have partial admission with little, if any, sacrifice of efficiency.

185. Conditions of Discharge.—Impulse wheels always discharge into the air. A reaction wheel may discharge either (*a*) into the air, (*b*) into a body of free water, or (*c*) into a suction tube.

When the discharge is into the atmosphere, the fall from the point of discharge to the tail race is lost. Such loss does not occur if the wheel is submerged, or if it discharges into a suction tube (provided the conditions are such that water fills the tube throughout its length).

The action of a suction tube depends upon atmospheric pressure. The pressure in the tube at the upper end (at Y, Fig. 104) is less than atmospheric by an amount depending upon the height of this point above the surface of the tail race. This decrease in the pressure at the point where the wheel discharges has the same effect upon the flow as an equal increase at X, the point of inflow. The fall z (Fig. 104) below the wheel,

which would be lost if the discharge occurred at the same point but under atmospheric pressure, is thus exactly compensated. If the wheel be placed lower or higher, the pressures at X and Y will change equally, so that the operation of the wheel will be unchanged. Of course z must not be so great that the pressure at Y is reduced to absolute zero.*

186. Girard Impulse Turbine.—Impulse turbines of the type known as the Girard have been extensively used in Europe, also to some extent in America. The flow may be either radial or axial, and admission may be either complete or partial.

The case of radial flow with complete admission may be represented as regards general arrangement by Fig. 95. The arrangement of a Girard turbine with axial flow is shown in Fig. 96.

Girard introduced the feature of ventilating the wheel passages by orifices for the free admission of air.

187. American Tangential Water Wheels.—The type of motor shown in Figs. 98 and 99 is common in America, especially in mountainous regions where high falls are utilized. The wheel vanes or buckets receive the water in a cylindrical stream from a nozzle, the axis of the stream being tangent to the circle described by a certain point in each bucket. Buckets have been made of many forms, but in most cases the "split" bucket is used, the stream being divided by a sharp edge into two streams which follow the opposite but similar bucket surfaces. Fig. 97 represents a section of such a bucket by a plane parallel to the axis of rotation and to the stream from the nozzle.

Although differing greatly in form from the turbines above mentioned, the tangential water wheel falls under the definition of turbine above given, and is of the impulse type with approximately axial flow.

188. Fourneyron Turbine.—This name is usually given to reaction turbines with radial outward flow. The discharge may be either into the air or into a body of water. In either

* A discussion of the suction tube is given in Art. 228.

case the general arrangement may be represented by Fig. 92. A suction tube cannot conveniently be used with this form of wheel.

189. Reaction Turbines with Inward Flow.—Wheels similar in principle to the Fourneyron but with inward flow have been designed by Thomson, Francis, and others. An inward-flow wheel with suction tube is shown in Fig. 105. This type is now quite generally known as the Francis turbine.

190. Jonval Turbine.—This is a reaction turbine with axial flow. The discharge may be either into the air, directly into the tail water, or into a suction tube (Fig. 104).

191. American Reaction Turbines. — The most common forms of reaction turbine used in America are of the mixed-flow type, having inward admission and axial discharge (Fig. 106).

In the United States the name turbine is usually confined to the wheels above called reaction turbines, while the "tangential" wheels described in Art. 187 are called water wheels.

192. Theory of Turbines. — The principles developed in Chapter XV furnish a basis for the theory of turbines and water wheels of all the foregoing types. In the following chapters will be given the outline of the theory for the impulse turbine, the American tangential water wheel, and the reaction turbine. A discussion will also be given of turbine pumps, of which the theory will be seen to be closely similar to that of reaction turbines.

The following definitions of available energy and efficiency apply to all types of water wheels and turbines acting as motors, but not to turbine pumps.

193. Available Energy.—The available energy of a stream of water * depends upon the weight of the water and the avail-

* Strictly this energy is possessed not by the water alone but by the system consisting of the water and the earth, being due to the gravitational attraction between the earth and the water. (Theoretical Mechanics, Art. 362.) The kinetic energy due to the velocity of flow in the stream is so small

able fall; thus Wh foot-pounds of potential energy are given up by W pounds of water in descending h feet. The object of a motor is to receive this from the water as mechanical energy, with as little loss by dissipation as possible.

The *available power* is the *energy available per unit time.* If the rate of discharge of the stream is q cu. ft. per sec. and the fall h ft., the available power is

$$wqh \text{ ft.-lbs. per sec.} = \frac{wqh}{550} \text{ H.P.}$$

194. Efficiency.—In estimating efficiency we may be concerned with the whole apparatus, including the pipe or channel leading to the motor, the motor proper, and the pipe (if any) leading from the motor to the tail race; or we may be concerned only with the efficiency of the motor proper. Again, we may consider the energy imparted to the motor by the water to be the useful effect, or we may regard the energy obtained from the motor in doing external work as the useful effect. In any case, the efficiency is defined by the equation

$$e = \frac{\text{energy utilized}}{\text{energy given up by water}};$$

but the numerator and denominator may each have two different values, according to the point of view as just explained.

Gross efficiency is computed by taking the external work done by the motor as "energy utilized."

Hydraulic efficiency is computed by taking energy imparted to the motor as "energy utilized."

These two efficiencies differ because of the energy dissipated by mechanical friction of the moving parts of the motor. Hydraulic efficiency corresponds to the "indicated" efficiency of a steam engine, since the indicated work is the work actually done on the piston by the steam pressure.

in comparison with the potential energy due to the fall, that it is neglected in estimating the available energy.

CHAPTER XVII.

THEORY OF IMPULSE TURBINE.

195. Given Dimensions and Data.*—In the theory of turbines certain dimensions and other data may be taken as known. Thus in an impulse turbine V_1, the velocity of outflow from the guides, is fixed independently of the construction of the wheel and of its speed of rotation. Since the discharge from the guides occurs under atmospheric pressure, V_1 depends only upon the fall to the point of outflow and the resistance occurring in the passages leading from the head race or reservoir. The angles A_1 and a_2 will also be taken as known. It may be seen that A_1 and $180°-a_2$ should both be small.

That a_2 should approach 180° is seen from the fact that the absolute velocity of outflow from the wheel (V_2) should be as small as possible, since the kinetic energy represented by this velocity is wholly lost. Since V_2 is the vector sum of u_2 and v_2 the more nearly opposite the directions of u_2 and v_2 are taken the smaller V_2 may become. This requirement is, however, limited by the fact that the smaller the angle $180°-a_2$ the smaller the cross-section of the wheel passages. Thus, let Fig. 94 represent a section of several vanes of an outward-flow turbine, and let d_2 denote the vertical dimension of the wheel passages at the point of outflow. If n is the total number of vanes, $2\pi r_2/n$ is the length of the portion of the wheel circumference corresponding to one passage, and the cross-section of a passage is approximately $(2\pi r_2 d_2 \sin a_2)/n$, so that

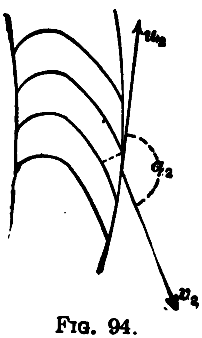


FIG. 94.

* For notation, see Art. 174.

approximately

$$f_2 = 2\pi r_2 d_2 \sin a_2.$$

That A_1 should be small is seen from formula (IV) (Art. 179), which gives the energy imparted to the wheel per unit time. Neglecting the term $u_2 s_2$, which is to be made small, and remembering that $s_1 = V_1 \cos A_1$, it is seen that for a given power u_1 must vary inversely as $V_1 \cos A_1$. Hence, to avoid an excessively high wheel speed, $V_1 \cos A_1$ should be as large as practicable. And since V_1 is fixed independently of the design of the wheel, $\cos A_1$ should be as great as practicable. This requirement is, however, limited by the fact that F_1 decreases as A_1 decreases, other dimensions remaining the same.

In the following theory the quantities taken as known are V_1, A_1, a_2, r_2/r_1 (which will be called c), $z_1 - z_2$. The discussion leads to a formula for efficiency, and to the determination of u_1 and a_1 for highest efficiency. These results are found to be independent of the remaining dimensions of the wheel.

(A) Girard Turbine with Radial Flow.

196. General Arrangement of Radial-flow Girard Turbine. —The general arrangement of a radial-flow Girard turbine with axis vertical and with complete admission is represented in Fig. 95. The wheel is placed as near the surface of the tail race as practicable. While the vertical widening of the wheel passages permits the particles of water to fall slightly during their passage through the wheel, the amount of this fall will be so small as to be unimportant, and will be neglected in the following theory. The figure shows a cylindrical gate so placed as to regulate the size of the guide outlets. The value of V_1 is not materially changed by varying the position of the gate, so that no serious loss of energy results from this method of regulating the supply of water.

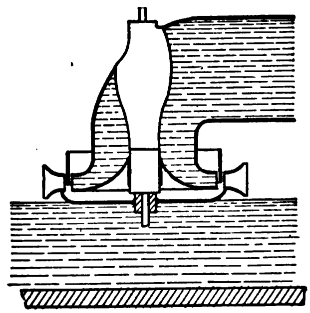


Fig. 95.

Girard turbines are often arranged for partial admission. The various forms which have been made, and the devices for regulating the admission of water to the buckets, will not be considered here, since these modifications do not affect the theory as here presented.

Governing, or the maintaining of a nearly uniform wheel speed in spite of fluctuations in the power consumed (or "load"), is accomplished by means of the regulating gate. This is connected by suitable mechanism with some form of centrifugal governor, so that an increase in the wheel speed causes a decrease in the gate opening.

197. Relation between Power and Wheel Speed.—Since the quantity of water used per second is independent of the speed of rotation, the efficiency is proportional directly to L, the energy imparted to the wheel by the water per second. It is necessary, therefore, to express L as a function of the velocity of rotation, and then consider what value of this velocity makes L a maximum. In the general formula (IV) (Art. 179) the variable velocities in the second member may all be expressed in terms of u_1 in the following manner.

We may write at once

$$u_2 = cu_1, \quad s_1 = V_1 \cos A_1.$$

To determine s_2 it is necessary to solve the problem of flow through the wheel, which may be done as follows:

From the vector relation between V_1, u_1 and v_1 (Fig. 93),

$$v_1^2 = V_1^2 + u_1^2 - 2V_1 u_1 \cos A_1. \quad . \quad . \quad . \quad . \quad (1)$$

The value of v_1 thus determined is to be substituted in formula (V) * for the purpose of determining v_2. Since in the present case $z_1 = z_2$ and $p_1 = p_2$, formula (V) takes the form

$$\frac{v_1^2 - u_1^2}{2g} - \frac{v_2^2 - u_2^2}{2g} = h'. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

It is necessary to estimate the value of h'.

* Art. 179.

The losses of energy by dissipation between the sections ($_1$) and ($_2$) are due mainly to two causes: (*a*) the interference of the inner edges of the wheel vanes with the stream from the guides, and (*b*) hydraulic friction within the wheel. To reduce the former as much as possible the edges of the vanes against which the stream impinges should be sharp, and the direction of these wheel vanes should be such as to agree with that of the relative velocity v_1. This direction of the wheel vanes can be adjusted to only one particular value of u_1, which should be that giving best efficiency, as yet unknown. For the purpose of this discussion, however, the direction of the vane may be regarded as varying with u_1 so as always to be parallel to v_1, leaving the actual value of the vane angle to be assigned when the best value of u_1 becomes known. Loss (*a*) will thus depend upon the relative velocity of the stream entering the wheel, while loss (*b*) will depend upon the relative velocity of flow through the wheel passages. At best only an approximate estimate of these losses is possible, and it will suffice to express the entire loss h' by a single term:

$$h' = k\frac{v_2^2}{2g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which k is a coefficient treated as constant, whose value cannot be accurately estimated apart from experiment. Equation (2) thus becomes

$$(1+k)v_2^2 = v_1^2 + (c^2-1)u_1^2, \quad . \quad . \quad . \quad . \quad (4)$$

which with (1) gives

$$(1+k)v_2^2 = V_1^2 - 2V_1u_1 \cos A_1 + c^2u_1^2. \quad . \quad . \quad . \quad (5)$$

Using this value of v_2, s_2 may be expressed in terms of u_1 and constants as follows:

$$s_2 = u_2 + v_2 \cos a_2 = cu_1 + \frac{\cos a_2}{\sqrt{1+k}}\sqrt{V_1^2 - 2V_1u_1 \cos A_1 + c^2u_1^2}. \quad (6)$$

We now have the values of s_1, s_2 and u_2, all in terms of u_1, for substitution in formula (IV). The result is

$$L=\frac{W}{g}\left[u_1V_1\cos A_1-c^2u_1^2 - \frac{c\cos a_2}{\sqrt{1+k}}u_1\sqrt{V_1^2-2u_1V_1\cos A_1+c^2u_1^2}\right]. \quad (7)$$

198. Condition for Maximum Power and Efficiency.—The mathematical condition for maximum L is $dL/du_1=0$. This leads to an equation of the fourth degree for determining u_1. The deduction of this equation and its solution in particular cases would involve a large amount of labor. In view of the imperfections in the foregoing theory, the following approximate treatment of the problem is probably as satisfactory a guide to design as the exact solution would be.

The greatest value of L results when the losses of energy are least. The loss which varies most with varying velocity is that due to the kinetic energy of the water leaving the wheel. Since V_2 is the vector sum of u_2 and v_2, it appears that when V_2 is small u_2 and v_2 will be nearly equal in magnitude. An approximate solution of the problem of maximum efficiency will therefore result from the assumption

$$v_2=u_2=cu_1. \quad \ldots\ldots\ldots \quad (8)$$

This value of v_2 substituted in (5) gives

$$V_1^2-2V_1u_1\cos A_1-kc^2u_1^2=0, \quad \ldots\ldots \quad (9)$$

which determines the value of u_1.

199. Value of Efficiency.—The hydraulic efficiency of the wheel is the ratio of L to the available energy per unit time, $WV_1^2/2g$. The efficiency for any velocity may thus be computed from the general value of L given by (7). By substitut-

ing the value of u_1 given by (9), the highest efficiency (according to the foregoing approximate solution) is found. The following convenient forms for the values of maximum power and efficiency are easily obtained by combining (7) and (9):

$$L_m=\frac{W}{g}[u_1V_1\cos A_1-c^2(1+\cos a_2)u_1^2], \quad . \quad . \quad (10)$$

$$e_m=2\left[\frac{u_1}{V_1}\cos A_1-c^2(1+\cos a_2)\frac{u_1^2}{V_1^2}\right], \quad . \quad . \quad (11)$$

in which u_1/V_1 must have the value given by (9).

200. Best Vane Angle.—The direction of a wheel vane at point of inflow should agree with that of the relative velocity v_1 possessed by the water leaving the guides, in order to prevent sudden deflection of the water and consequent loss of energy and decrease of efficiency. Since the angle A_1 is fixed, the vector triangle for u_1, v_1, V_1 has all its angles determined when u_1/V_1 is known. In fact equations (II') * give

$$\text{cotan } a_1=\text{cotan } A_1-\frac{u_1}{V_1}\text{cosec } A_1. \quad . \quad . \quad . \quad (12)$$

Simple approximate rule.—If k be assumed zero, equation (9) gives

$$V_1=2u_1\cos A_1,$$

showing that the triangle whose sides are u_1, v_1, V_1 is isoscelcs, v_1 being equal to u_1, and therefore

$$a_1=2A_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12a)$$

This is often given as the rule for best value of a_1.

* Art. 179.

(B) Girard Turbine with Axial Flow.

201. General Arrangement.—A Girard turbine with vertical axis and axial flow is represented in general arrangement in Fig. 96. The wheel discharges as near the surface of the tail-water as practicable, and in order that it may act strictly as an impulse wheel the buckets should be provided with orifices for the free admission of air. The features which distinguish this case from the preceding, as regards the theory, are the equality of r_1 and r_2, and the inequality of z_1 and z_2.

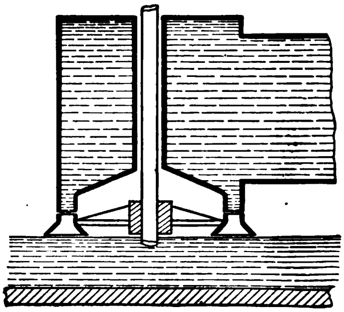


Fig. 96.

202. Best Velocity.—Following the same general method as in the case of radial flow, formula (V) now takes the form

$$\frac{v_1^2}{2g}-\frac{v_2^2}{2g}+z_1-z_2=h', \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

which replaces equation (2). Making the same assumption as to the value of h' as in the preceding case, and also assuming that $u_2=v_2$ for highest efficiency, we are led to the equation

$$V_1^2-2uV_1\cos A_1-ku^2+2g(z_1-z_2)=0, \quad . \quad . \quad (14)$$

in which u is written for each of the equal velocities u_1 and u_2. This replaces equation (9).

In many cases z_1-z_2 is a small fraction of the total fall and may be neglected; but for very low falls it must be retained.

203. Efficiency.—The available energy in this case includes that due to the fall within the wheel, z_1-z_2; therefore when this is an important fraction of the total fall, the general formula for efficiency is

$$e=\frac{L}{W\left(\frac{V_1^2}{2g}+z_1-z_2\right)},$$

and the maximum efficiency is found by using the value of L corresponding to the best speed as computed from (14). This may be written in the form

$$L_m = \frac{W}{g}[uV_1 \cos A_1 - (1 + \cos a_2)u^2]. \quad . \quad . \quad . \quad (15)$$

Equations (14) and (15) correspond to (9) and (10) of the preceding case.

204. Best Vane Angle.—When $z_1 - z_2$ is not negligible, the best value of the vane angle depends not merely upon the ratio of u and V_1 (as in the preceding case), but upon their absolute values, since equation (14) does not give a value of u/V_1 which is independent of the actual value of V_1. The best value of the vane angle cannot, therefore, be determined except by assuming the ratio of $z_1 - z_2$ to $V_1^2/2g$. Strictly speaking, then, a given wheel cannot work with equal efficiency under different falls. This consideration is of little importance if the fall exceeds a very few feet.

When u and V_1 are known, a_1 may be computed from equaion (12).

EXAMPLES.

1. Take data as follows for an impulse turbine with radial outward flow: $A_1 = 20°$, $a_2 = 160°$, $r_2/r_1 = 1.2$. Assuming $k = 0$, determine the best speed, best vane angle, and highest efficiency.

Solution.—The best speed is determined from equation (9), in which $c = 1.2$, $k = 0$, giving $u_1/V_1 = .532$. From equation (11) the maximum efficiency is .951. The best value of a_1, determined from equation (12) with the above value of u_1/V_1, is 40°.

2. Solve Ex. 1 assuming $k = .2$.

Ans. $u_1/V_1 = .495$; $e = .890$; $a_1 = 37° \ 35'$.

3. With data as in Ex. 1, determine what value of k reduces the greatest efficiency to .80, and find the corresponding values of u_1/V_1 and a_1.

Solution.—This may be solved by trial. Trying $k = .5$, there result the values $u_1/V_1 = .456$, $e = .820$, $a_1 = 35° \ 15'$. Comparing with preceding results it may be estimated that the required value of k is about .6. Solving with this value, $u_1/V_1 = .442$, $e = .80$, $a_1 = 34° \ 30'$.

4. Instead of the assumption $u_2 = v_2$, upon which the above solution for maximum efficiency is based, it is sometimes assumed that the best velocity is that which makes $A_2 = 90°$. Show that this leads to an equation like (9) with k' substituted for k, where $k' = (1+k) \sec^2 a_2 - 1$.

5. Show that the assumption $A_2 = 90°$ leads to $e = 2(u_1/V_1) \cos A_1$.

6. Take data as in Ex. 1 and solve on the assumption $A_2 = 90°$, for $k=0$, .2, and .6. Estimate what value of k will give an efficiency of .80.

Ans. For $k = 0$, $u_1/V_1 = .507$, $e = .952$, $a_1 = 38° \, 19'$.
" $k = .2$, $u_1/V_1 = .471$, $e = .885$, $a_1 = 36° \, 5'$.
" $k = .6$, $u_1/V_1 = .422$, $e = .793$, $a_1 = 33° \, 28'$.

7. Assume dimensions as in Ex. 1, and $k = .6$. If the quantity of water to be utilized is 16 cu. ft. per sec., and the total fall is 600 ft., 10 per cent of which is lost by friction in the supply-pipe, determine the values of F_1 and V_1. *Ans.* $V_1 = 186.4$. $F_1 = 16/186.4 = .0858$ sq. ft.

8. In Ex. 7 determine highest efficiency and power.
Ans. $e = .80$, $L = 432{,}000$ ft.-lbs. per sec. H.P. $= 785$.

9. An impulse turbine with vertical axis and axial flow is to utilize a fall of 11 ft. Take $z_1 - z_2 = 1$ ft., $A_1 = 26°$, $a_2 = 156°$, and assume that the loss of head in the supply-pipe and guide-passages is negligible. Determine u/V_1, e, and a_1, if $k = 0$.
Ans. $V_1 = 25.4$, $u = 15.5$, $u/V_1 = .61$, $e = .935$, $a_1 = 56° \, 40'$.

10. With data of Ex. 9, find what value of k will make $e = .80$, and determine the corresponding values of u/V_1 and a_1.
Ans. $k = .65$, $u = 13.1$, $u/V_1 = .515$, $e = .80$, $a_1 = 48° \, 50'$.

11. In Ex. 10, if the wheel is to use 80 cu. ft. of water per sec., what must be the value of F_1? What H.P. will be realized?

CHAPTER XVIII.

THE TANGENTIAL WATER WHEEL.

205. Introductory Illustration.—Let Fig. 97 represent a section of a vane or bucket receiving a jet of water from a nozzle, and let the vane be moving in the same direction as the jet, the velocity of the water being V_1 and that of the vane u. The vane consists of two symmetrical parts coming together in a sharp edge. This edge divides the jet into two streams which follow the two parts of the vane, being gradually deflected and finally discharged at its outer edges

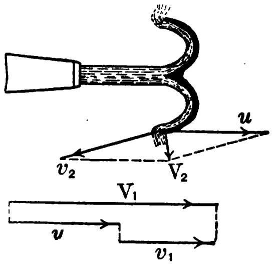


FIG. 97

The force exerted upon the vane by the jet and the work done by this force per unit time may be computed by the methods explained in Chapter XIV. With the notation described in Art. 174, it is seen that $u_2 = u_1 = u$, $A_1 = 0$, $a_1 = 0$, $v_1 = V_1 - u$, while a_2 is determined by the tangent to the vane curve at the outer edge.

The force constantly exerted upon the vane is equal to the change in the momentum of the water striking it in one second (Art. 158). If the two partial streams are equal, symmetry shows that the direction of the resultant force coincides with that in which the vane is moving; we therefore resolve the momentum in this direction. The momentum of W lbs. of water before striking the vane is WV_1/g, while that of an equal quantity leaving the vane is WV_2/g. The component of the latter in the direction of the motion of the vane is

$$\frac{W}{g}V_2 \cos A_2 = \frac{W}{g}(u + v_2 \cos a_2).$$

Hence the force exerted upon the vane is

$$P=\frac{W}{g}(V_1-u-v_2\cos a_2).$$

If frictional resistance be neglected, the relative velocity will remain constant during the flow over the vane, so that

$$v_2=v_1=V_1-u,$$

and

$$P=(1-\cos a_2)\frac{W}{g}(V_1-u).$$

The work done per second by the force P is $L=Pu$; or on the assumption of no frictional loss,

$$L=(1-\cos a_2)\frac{W}{g}(V_1-u)u.$$

For a given value of W this work has its greatest value when $u=V_1/2$, the value being

$$L=\frac{1-\cos a_2}{2}\cdot\frac{WV_1^2}{2g}.$$

If $a_2=180°$, this reduces to $WV_1^2/2g$, showing that the kinetic energy of the water striking the vane is all given up to the vane.

206. General Features of Tangential Water Wheel. — The tangential water wheel is designed to realize as nearly as possible the ideal conditions assumed in the foregoing illustration. With this object buckets are mounted upon the rim of a wheel, and a cylindrical jet is thrown against these in such a way that each bucket, while receiving the jet, is moving as nearly as practicable in the same direction as the jet. If the water could be received and gradually deflected by each bucket without dissipation of energy until its relative velocity became equal and opposite to the velocity of the bucket, the energy of the jet would be wholly utilized. The practical limitations which pre-

vent a close approximation to this ideal case are indicated in the following discussion.

The general characteristics of wheels of this type are shown in Figs. 98 and 99. The fundamental features which distin-

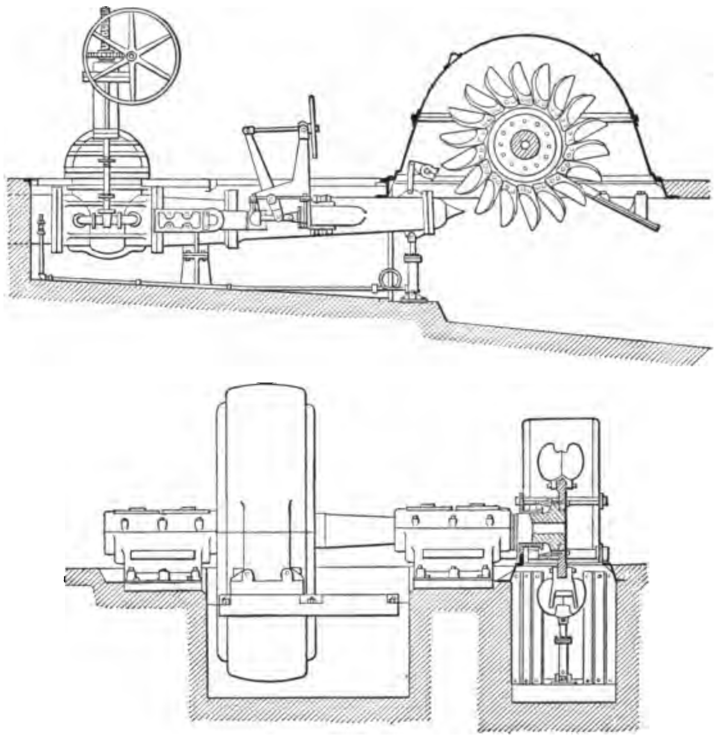


FIG. 98. Two views of a Doble water wheel generating 8000 H. P. under a working head of 1528 ft.

guish these motors from those of the Girard type are the use of circular nozzles for "guide-passages," and the double or "split" character of the buckets. Tangential wheel buckets have been made of various forms, the relative merits of which will not be discussed. Separate views of wheel-runners of two different makes are represented in Figs. 100 and 101.

The relation of the buckets to the jet is shown in Fig. 102. Here the bucket A is receiving the jet centrally, while A' is just

entering the jet and A'' is nearly in the position where the jet ceases to strike it. While receiving the jet each bucket thus has a range of motion $A'A''$, the angle between its direction of motion and that of the jet varying by the amount $A'CA''$. The following theoretical discussion will refer to conditions as

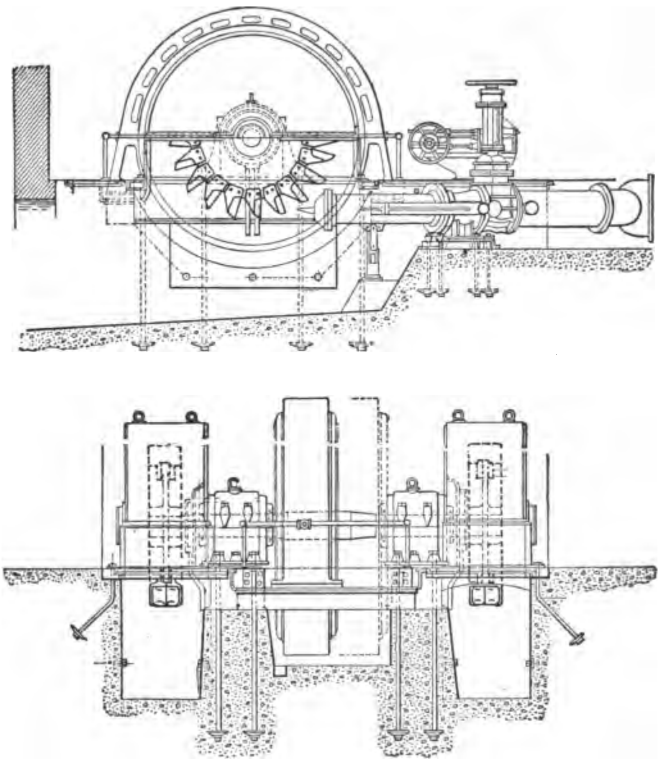


FIG. 99. Two views of a Pelton unit consisting of two water wheels with a total capacity of 7500 H. P. under a working head of 872 ft.

they exist for a mean position of the bucket. A horizontal section through the bucket and jet in this mean position is shown in Fig. 102 (B). The motion of the water *relative to the bucket* is not, however, in this plane, but has an upward component because of the downward component of the velocity of the bucket. This is shown by the vector triangle at (C). The angle a_2, between the velocity of the bucket (u) and the relative

Fig. 100.
Pelton Water-wheel Runner.

Fig. 101.
Doble Water-wheel Runner.

Fig. 107.
Runner of Francis Turbine
(Platt Iron Works Company).

Fig. 108.
Runner of Victor Turbine
(Platt Iron Works Company).

velocity of the water leaving the wheel (v_2), is not shown in its true size in the figure, since neither u nor v_2 is parallel to the plane of the sectional view (B). A mean value, depending upon the construction of the bucket, may be assumed for the angle a_2, and the form of the bucket should be such as to make this angle as near 180° as will permit the water to clear the following bucket.

207. Best Wheel Speed.—The theory of the tangential water-wheel may be expressed in substantially the same form as that of the Girard turbine, if the same notation is employed.

The relative velocity v_1 of the water just about to strike the bucket is computed, for a given value of u, from the vector triangle for u, v_1, V_1. This triangle lies in a vertical plane, and is shown at (C), Fig. 102. Thus

$$v_1^2 = V_1^2 + u^2 - 2V_1 u \cos A_1. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

To compute v_2 the general formula (V) * is to be applied; and since practically $u_1 = u_2$ and $z_1 = z_2$, the equation becomes

$$\frac{v_1^2}{2g} - \frac{v_2^2}{2g} = h'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The loss of head h' includes the loss in friction of the stream while flowing over the bucket surface, the loss due to impingement of the jet against the sharp edge or "splitter," and the loss due to the interference of the edge of the bucket with the jet before it is in position to receive the whole stream. These losses all vary with the relative velocity. We may adopt the usual assumption that the loss is proportional to the square of this velocity, and express it in the same way as in the theory of the Girard turbine:

$$h' = k\frac{v_2^2}{2g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

* Art. 179.

Equations (1), (2) and (3) then give

$$(1+k)v_2^2 = V_1^2 - 2V_1 u \cos A_1 + u^2, \quad . \quad . \quad . \quad (4)$$

from which v_2 may be found.

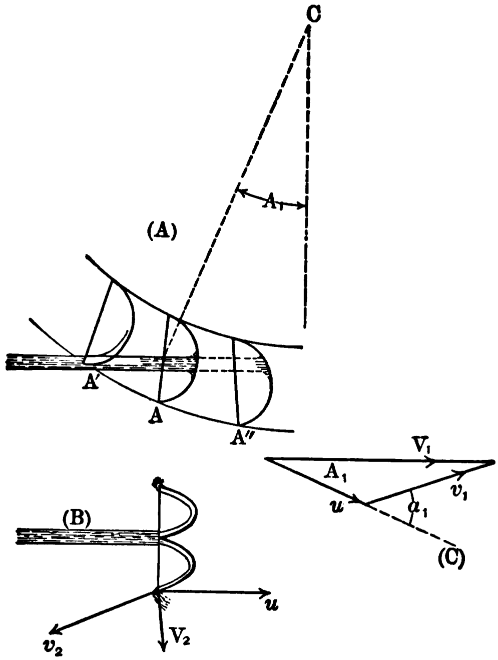


FIG. 102.

The next step would be to substitute in the general formula (IV) for L

$$s_1 = V_1 \cos A_1,$$

$$s_2 = u + v_2 \cos a_2,$$

thus expressing L in terms of u. The resulting equation would be identical with (7) of Chapter XVII, with $c=1$.

208. Approximate Solution for Best Velocity.—The application of the mathematical condition for maximum L leads to an equation of the fourth degree for determining u. As a simpler and sufficiently exact solution the same approximate

assumption may be made as in the case of the Girard turbine (Chapter XVII).

The unutilized energy (per pound of water) is made up of the "frictional" loss $k(v_2^2/2g)$ and the kinetic energy possessed by the water as it leaves the wheel, $V_2^2/2g$. While the former is the more important, the latter probably varies more rapidly as the speed varies from that giving greatest efficiency, so that the highest efficiency will be obtained when V_2 is small, and this will be when v_2 and u are nearly equal. Assuming

$$v_2=u, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

equation (4) becomes

$$V_1^2-2V_1u\cos A_1-ku^2=0, \quad . \quad . \quad . \quad . \quad (6)$$

from which u/V_1 may be computed.

209. Efficiency.—Taking as the available energy the kinetic energy of the jet from the nozzle, the hydraulic efficiency is

$$e=\frac{L}{WV_1^2/2g}=\frac{2u(s_1-s_2)}{V_1^2}. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Putting $v_2=u$, $s_2=u+v_2\cos a_2=u(1+\cos a_2)$, the highest efficiency according to the foregoing solution is

$$e_m=2\left[\frac{u}{V_1}\cos A_1-(1+\cos a_2)\frac{u^2}{V_1^2}\right], \quad . \quad . \quad . \quad (8)$$

in which u/V_1 must have the value given by equation (6).

210. Form of Bucket Surface.—While a bucket is receiving the jet it moves through a certain distance in a direction oblique to that of the jet and turns through a small angle. The point at which the jet strikes it, and the direction of the relative motion of the impinging water, are therefore variable, and the form of the bucket surface should conform to this variation. Without entering into a detailed study, it may be said that the form should be such as to deflect the water as gradually as possible throughout its entire passage over the bucket surface, and to make the direction of the relative velocity of outflow as

nearly opposite to the velocity of the bucket as practicable. The varying aspect presented by the bucket to the stream seems to require a surface of double curvature if sudden changes of velocity are to be avoided. The dividing edge should, of course, be as sharp as practicable. The limitation of the direction of outflow imposed by the necessity that the water leaving one bucket shall clear the next one has already been mentioned.

211. Comparison with Theory of Girard Turbine.—The formulas above deduced for the tangential water wheel are seen to be identical in form with those obtained for the radial-flow Girard turbine except that $c=1$. They may also be obtained from the formulas applying to the axial-flow Girard wheel by putting $z_1=z_2$.

212. Conditions Favorable for Use of Tangential Water Wheel.—In general it may be said that wheels of this type may be used to advantage wherever the fall utilized is great and the supply of water relatively small, and are at a decided disadvantage only when the fall is but a few feet, or when the supply of water is great. They are in efficient operation under falls as great as 2500 ft.

213. Actual Efficiencies.—While accurate tests are rare, it is probable that the best tangential water wheels give efficiencies as great as are obtained from any type of turbine or water wheel. Hydraulic efficiencies as great as 80 to 85 per cent under heads up to 1500 or 2000 ft. are doubtless not uncommon. The extremely high efficiencies sometimes claimed by manufacturers must be regarded with suspicion.

214. Regulation.—An important problem in connection with the practical working of a water wheel is the regulation of the discharge from the nozzle in order to vary the power output of the wheel, or to conform to variations in the supply of water. The partial closing of a valve of any ordinary form placed in the supply-pipe or nozzle would cause a serious loss of energy. In some cases two or more nozzles are placed at different points

of the circumference of the wheel, any combination of which may be used as required.

The *needle nozzle* (Fig. 103) is a regulating nozzle of peculiar

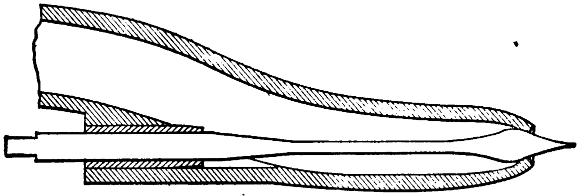


FIG. 103.

form, designed to vary the size of the jet at pleasure with little loss of efficiency. The conical valve or "needle" can be moved parallel to the axis of the jet, so as to leave any desired amount of opening between the needle and the nozzle tip. If carefully made, this device accomplishes the object of regulation with little loss of energy very satisfactorily.*

215. Governing.—Another important problem is that of maintaining a uniform speed of rotation when the load fluctuates. This is usually accomplished by deflecting the jet so that a varying portion of it strikes the buckets, the deflection being permitted by a joint in the nozzle pipe, and the movement being controlled by some form of centrifugal governor.

Governing has also been successfully accomplished by means of the needle nozzle, the needle being actuated by the governor so that the nozzle opening varies with any variation in the speed.

EXAMPLES.

1. If $A_1 = 20°$ and $a_2 = 160°$, determine best value of u/V_1 and highest efficiency assuming $k = 0$. *Ans.* $u/V_1 = .532$. $e = .966$.

2. Solve Ex. 1 assuming $k = .5$. *Ans.* $u/V_1 = .473$. $e = .860$.

3. In the same case, if the maximum efficiency is .80, what is the value of k, and what is the best wheel velocity?

4. If the effective head at the nozzle is 600 ft. and the wheel is to make 700 R.P.M., what should be its diameter?

5. If the nozzle throws a jet 1.5 inches in diameter, determine the power.

* The needle nozzle is covered by U. S. patent.

CHAPTER XIX.

THEORY OF THE REACTION TURBINE.

216. General Features of Reaction Turbines.—The fundamental characteristic of a reaction turbine is the fact that the wheel passages are completely filled by the streams flowing through them. The pressure within these passages, therefore, is not determined by contact of the stream with air as in the impulse wheel, but in general varies continuously between the points of admission and discharge.

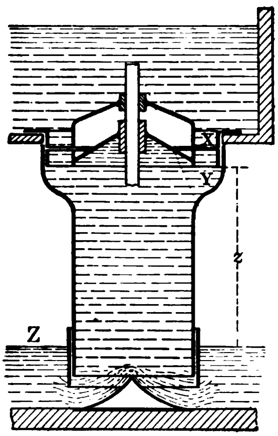


Fig. 104.

The general arrangement of a reaction turbine with radial outward flow (the Fourneyron type) is shown in Fig. 92, while the case of axial flow (Jonval) is represented in Fig. 104, and that of inward flow (Francis type) in Fig. 105. The turbine shown in Fig. 105 does not closely resemble the original design of Francis, being adapted to a much greater fall.

A design quite generally adopted in American practice is that of mixed flow (inward admission and axial discharge). This is represented in Fig. 106. There is no sharp distinction between this and the form represented in Fig. 105.

Turbine runners of two forms are shown in Figs. 107 and 108, the former being of the inward-flow or Francis type, the latter of the mixed-flow type similar to those in Fig. 106.

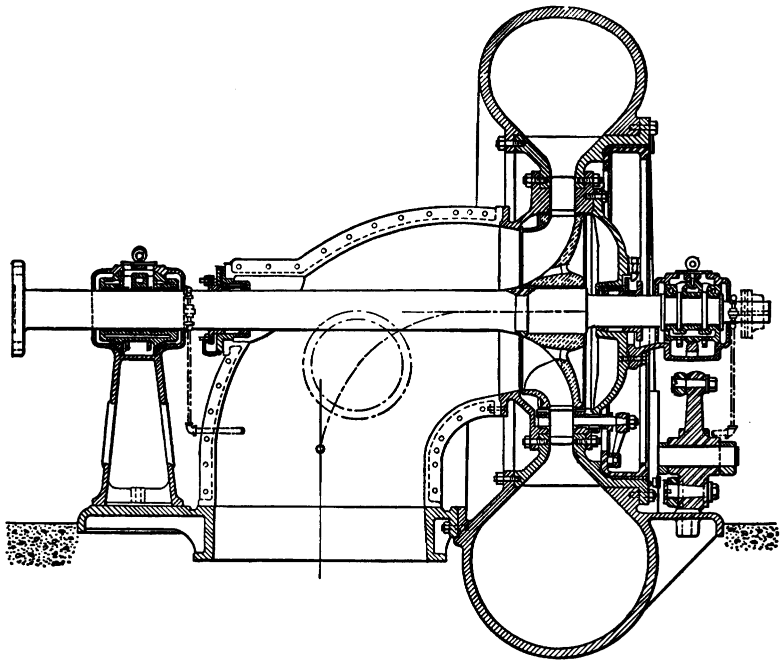


FIG. 105. Modern Francis turbine, of capacity 5000 H. P., under a fall of 320 ft. Built by the Allis-Chalmers Company.

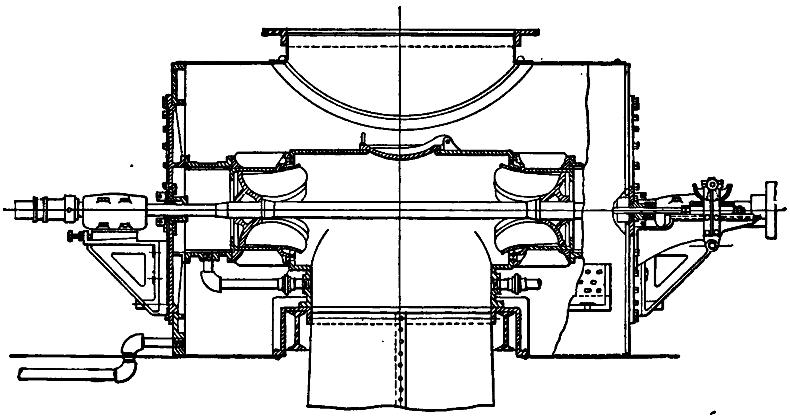


FIG. 106. A pair of turbines installed by the I. P. Morris Co. The working head is 82 ft., the speed 600 revolutions per minute, and the power generated 750 H.P.

The following theory as at first presented refers to the Fourneyron or outward-flow turbine, as represented in Fig. 92. The main features of the theory hold for reaction turbines of other forms, the points of difference being indicated later.

217. Data and Notation.—In the following discussion of reaction turbines the notation will for the most part be the same used in the foregoing theory of impulse wheels and explained in Art. 174. In addition the following symbols will be used.

Assuming the wheel to discharge into the atmosphere,* let h denote the total fall from surface of supply reservoir or head-race to the place of discharge from the wheel, and let h' denote the total loss of head (i.e., the part of h that is not utilized).

In applying to the reaction turbine the general formulas deduced in Chapter XV, the special condition must be introduced that the cross-section of the stream within the wheel is everywhere fixed, not varying with the wheel speed nor with the velocity of flow. Thus in the equation of continuity (formula (I'), Art. 179) both F_1 and f_2 are constant, so that v_2 and V_1 are in a constant ratio. That is,

$$v_2 = c'V_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which $c' = F_1/f_2 = \text{constant}.$†

The following data will be taken as known: h, A_1, a_2, $c' = F_1/f_2$, $c = r_2/r_1$.

218. Relation between Wheel Speed and Rate of Flow through Wheel.—The velocity of flow in every section of the stream is

* If the wheel is submerged, h will mean the fall from surface of supply water to surface of waste water; the same formulas will then apply as in the case of discharge into the air. The effect of a suction-tube is considered in Art. 228.

† It will be noticed that the condition $F_1/f_2 = \text{constant}$ in the reaction turbine replaces the condition $p_1 = p_2$ in the impulse turbine. Formula (V), which was used in determining the flow through the wheel in the previous case, is of use in the present problem only in determining the relation between p_1 and p_2.

determined if that in any one section is fixed. Let all such velocities be expressed in terms of V_1, and let it be required to determine the relation between V_1 and the wheel speed u_1.

From the above meanings of h and h' it is seen that $h-h'$ is the utilized head, and that the energy received by the wheel from the water per unit time is

$$L = W(h-h'). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence from formula (IV), Art. 179,

$$W(h-h') = \frac{W}{g}(u_1 s_1 - u_2 s_2),$$

or

$$g(h-h') = u_1 s_1 - u_2 s_2. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The second member of this equation can be expressed in terms of V_1 and u_1. Thus

$$u_2 = cu_1; \quad s_1 = V_1 \cos A_1;$$

$$s_2 = u_2 + v_2 \cos a_2 = cu_1 + c'V_1 \cos a_2;$$

so that (3) may be written

$$g(h-h') = (\cos A_1 - cc' \cos a_2) V_1 u_1 - c^2 u_1^2. \quad . \quad . \quad (4)$$

It is now necessary to express the value of h'.

The most important losses of head may be expressed by two terms. One of these represents the "frictional" loss occurring throughout the stream, which by the usual rule of hydraulics is expressed as proportional to the square of the velocity of flow. Since the velocities in all sections vary in the same ratio, we may assume

$$k\frac{v_2^2}{2g} = \text{total frictional loss of head,}$$

k being a coefficient whose value depends upon the dimensions and character of the entire series of passages, and which will

be treated as constant. The other important loss of head is that represented by the kinetic energy of the water leaving the wheel. Combining this with the frictional loss, we may write

$$h'=k\frac{v_2^2}{2g}+\frac{V_2^2}{2g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From the vector triangle for u_2, v_2, and V_2,

$$\begin{aligned} V_2^2 &= u_2^2+v_2^2+2u_2v_2 \cos a_2 \\ &= c^2u_1^2+c'^2V_1^2+2cc'u_1V_1 \cos a_2, \end{aligned}$$

so that (5) may be written in the form

$$2gh'=(1+k)c'^2V_1^2+2cc' \cos a_2 \cdot V_1u_1+c^2u_1^2. \quad . \quad . \quad . \quad (6)$$

Eliminating h' between (4) and (6),

$$(1+k)c'^2V_1^2+2 \cos A_1 \cdot V_1u_1-c^2u_1^2=2gh. \quad . \quad . \quad (7)$$

By solving this equation V_1 may be determined for any value of u_1. For a given wheel, F_1 being fixed in value, the rate of discharge varies as V_1, being given (when V_1 is known) by the equation

$$W=wF_1V_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

219. Power and Efficiency for any Wheel Velocity.—Expressing $u_1s_1-u_2s_2$ in terms of u_1 and V_1 as in Art. 218, the general formula (IV) may be written

$$L=\frac{W}{g}[(\cos A_1-cc' \cos a_2)V_1u_1-c^2u_1^2]. \quad . \quad . \quad . \quad (9)$$

From this, by using the value of V_1 given by (7), the power corresponding to any value of u_1 may be computed, provided F_1 is known so that W can be determined.

The *efficiency*, however, is independent of the value of F_1. For W lbs. of water the available energy is Wh foot-pounds, hence

$$e=\frac{L}{Wh}=\frac{1}{gh}[(\cos A_1-cc' \cos a_2)V_1u_1-c^2u_1^2], \quad . \quad (10)$$

in which V_1 must have the value given by (7) for any value of u_1.

220. Best Speed and Highest Efficiency.—The speed giving maximum efficiency may be determined by applying the condition $de/du_1=0$. The direct method of procedure would be to solve equation (7) for V_1 and substitute its value in (10), thus expressing e explicitly in terms of the single variable u_1 before differentiating. The following indirect method is less laborious.

Differentiating (10),

$$gh\frac{de}{du_1}=(\cos A_1-cc'\cos a_2)u_1\frac{dV_1}{du_1}$$
$$+(\cos A_1-cc'\cos a_2)V_1-2c^2u_1=0.$$

Differentiating (7),

$$[(1+k)c'^2V_1+\cos A_1\cdot u_1]\frac{dV_1}{du_1}+\cos A_1\cdot V_1-c^2u_1=0.$$

Eliminating dV_1/du_1 between these two equations, and reducing,

$$(1+k)c'^2(\cos A_1-cc'\cos a_2)V_1^2-2(1+k)c^2c'^2V_1u_1$$
$$-c^2(\cos A_1+cc'\cos a_2)u_1^2=0. \qquad (11)$$

This equation must be satisfied when e is a maximum. Combining it with (7), which is always true, the values of u_1 and V_1 corresponding to maximum efficiency may be found. The solution may be expressed as follows:

Equation (11) determines * the value of V_1/u_1. Calling this α, and substituting $V_1=\alpha u_1$ in (7),

$$[(1+k)c'^2\alpha^2+2\cos A_1\cdot\alpha-c^2]u_1^2=2gh, \quad . \quad . \quad . \quad (12)$$

from which may be computed the value of u_1 giving maximum efficiency.

* Two values of V_1/u_1 are given by (11); which value corresponds to the practical problem is seen in any particular case after substitution in (7).

221. Best Angle of Wheel Vane.—The direction of the wheel vane at the point of inflow should agree with that of the relative velocity of outflow from the guide passages, in order to prevent sudden deflection of the water and consequent loss of energy. The value of a_1 is fixed by that of V_1/u_1, and may be determined in the usual manner by solving the vector triangle whose sides are u_1, V_1, v_1. Equation (12) of Art. 200 may be employed:

$$\cot a_1 = \cot A_1 - \frac{u_1}{V_1} \operatorname{cosec} A_1. \quad . \quad . \quad . \quad (13)$$

Since u_1/V_1 varies with the wheel velocity, a_1 also varies with u_1. The value corresponding to maximum efficiency should govern the design of the wheel.

In the foregoing theory the loss of head due to sudden deflection of the stream entering the wheel is neglected, which is equivalent to assuming that the vane angle varies with the speed of rotation so as always to agree with a_1. This is allowable in solving for maximum efficiency, since the vane angle is to be made to agree with the value of a_1 which is finally found as the result of the solution.

For a given wheel, however, there is only a single value of u_1/V_1 which makes the assumption of no sudden deflection of the stream true, and the loss of head due to this cause should be taken into account in an accurate solution of the problem of flow through the wheel for any rotation speed. In spite of this, however, it is probable that equation (7) shows approximately the way in which V_1 varies with u_1 in an actual wheel, for a considerable range of values of u_1.

222. Pressure at Entrance to Wheel.—If h_1 denotes the fall from head-race to point of outflow from guides, and H' the loss of head between these points, the equation of energy gives

$$\frac{p_1}{w} = h_1 - \frac{V_1^2}{2g} - H',$$

p_1 being reckoned from atmospheric pressure as zero.

EXAMPLES.

Given $A_1=28°$, $a_2=158°$, $r_1=3.375$ ft., $r_2=4.146$ ft., $F_1=6.537$ sq. ft., $f_2=7.687$ sq. ft., $h=12.8$ ft.

1. Assuming $k=.5$, determine best wheel velocity, best value of vane angle, and highest efficiency.

Ans. $u_1=.50\sqrt{2gh}=14.4$ ft. per sec.; $e=.71$; $a_1=61° 45'$.

2. By trial of different values of k, determine what value gives a maximum efficiency of 80 per cent. *Ans.* $k=.25$ very nearly.

223. Introduction of Ratios of Velocities.—Let m denote the velocity equivalent to the fall h, i.e.,

$$m^2=2gh;$$

then it is seen that the main equations reached by the above theory really involve the *ratios* of the three velocities u_1, V_1, m, rather than their actual values. Thus, comparing different cases in which h has different values, if u_1, V_1 and m are changed in the same ratio, equations (7), (10), and (11) are unchanged. It is instructive to write the equations so as to involve the ratios of the three velocities explicitly.

Let $u_1/m=x$, $V_1/m=y$. Then equations (7), (10), and (11) become

$$(1+k)c'^2y^2+2\cos A_1\cdot yx-c^2x^2=1, \quad . \quad . \quad . \quad \text{(A)}$$

$$e=2(\cos A_1-cc'\cos a_2)yx-2c^2x^2, \quad . \quad . \quad . \quad \text{(B)}$$

$$(1+k)c'^2(\cos A_1-cc'\cos a_2)y^2-2(1+k)c^2c'^2yx$$
$$-c^2(\cos A_1+cc'\cos a_2)x^2=0. \quad \text{(C)}$$

Of these equations (A) and (B) are general, while (C) holds only for maximum efficiency.

The values of the rate of discharge and of the power depend not merely upon ratios of velocities, but upon their actual values. They may be written as follows:

$$q=F_1V_1=F_1my, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(D)}$$

$$L=Whe=wqhe=wF_1myhe=\frac{wF_1m^3}{2g}ey. \quad . \quad . \quad . \quad \text{(E)}$$

224. Graphical Representation.—If x and y are taken as rectangular coordinates, equation (A) represents a curve which shows graphically the way in which the rate of discharge varies with the wheel velocity for any constant fall; for when m in constant, V_1 is proportional to y and u_1 to x.

The value of e for any value of x and the corresponding value of y may be computed from (B). If a curve be drawn with e as ordinate and x as abscissa, it shows graphically the

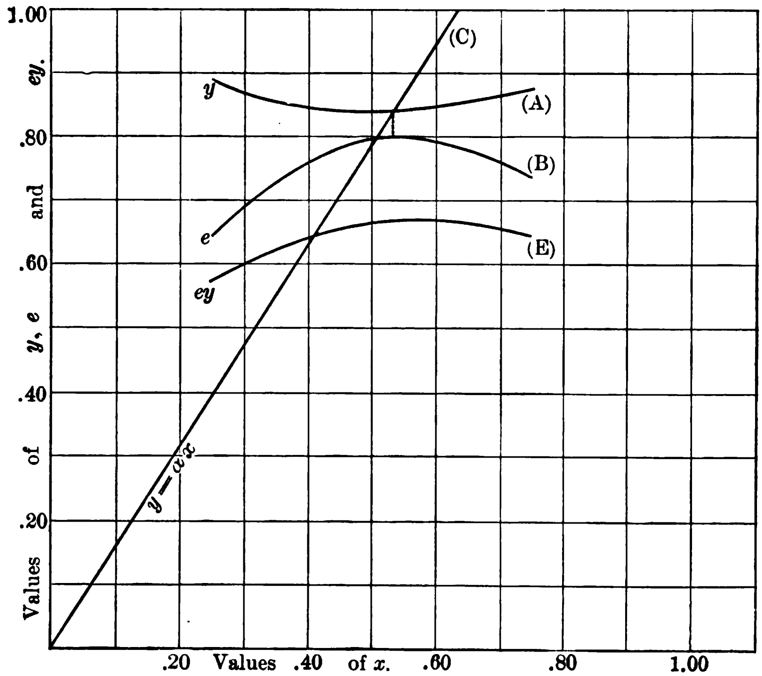


FIG. 109.

variation of the efficiency with the wheel speed when h is constant.

The variation of the power with the wheel velocity when h is constant is shown by a curve determined by taking as abscissas values of x and as ordinates values of ey, the variable factor in the value of L as given by (E).

These three curves, for the data of the examples after Art. 222, with $k=0.25$, are shown in Fig. 109.

Equation (A) represents an hyperbola, while (C) represents two straight lines, one of which intersects the hyperbola in a point whose coordinates are the values of x and y for maximum efficiency.

These curves cannot, of course, be regarded as accurately representing the results actually to be expected from a turbine of given dimensions. The uncertain element in the theory is the value of the loss of head h'; in particular the remarks made in Art. 221 regarding loss at entrance must be borne in mind.

It should be noticed that the value of e given by equation (B) involves no constants except dimensions of wheel and guides, and that this equation is not affected by the uncertainty in the value of h'. If the true relation between x and y were known (as, for example, by experiment), the efficiency curve could be correctly drawn from equation (B).

225. Approximate Solution for Maximum Efficiency.—Reasoning as in the case of the impulse turbine (Art. 198), it appears that the greatest efficiency will result when the absolute velocity of outflow from the wheel has a small value. The assumption $v_2 = u_2$ will therefore give an approximation to the solution for maximum efficiency. The more common assumption, however, is that $A_2 = 90°$, or

$$V_2 \cos A_2 = u_2 + v_2 \cos a_2 = 0. \quad . \quad . \quad . \quad . \quad (14)$$

Either of these assumptions leads to a simple equation in place of (C). Thus, from (14),

$$cu_1 + c' \cos a_2 \cdot V_1 = 0,$$

or

$$cx + c' \cos a_2 \cdot y = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (C')$$

which replaces (C). The solution is otherwise unchanged.

In the case represented by the curves in Fig. 109 the approximate solution for maximum efficiency gives practically the same result as the exact solution. The value of y/x or α given by (C′) is almost identical with that given by (C); both these equations are represented by the straight line marked (C), and

the intersection of this line with the curve (A) gives the values of x and y corresponding to maximum efficiency.

EXAMPLES.

1. Using the results of the exact solution of Ex. 1, Art. 222, determine the value of A_2. *Ans.* $92° 30'$.

2. With same data, assume $A_2 = 90°$, and determine x, y, e, a_1.
Ans. $x = .51$, $y = .79$, $e = .71$, $a_1 = 62° 50'$.

226. Cases of Inward, Axial, and Mixed Flow.—All these cases are covered by the foregoing theory, so long as the wheel discharges directly into the air or below the surface of the tail-race. With axial flow $c = 1$, with inward flow or mixed inward and axial flow $c < 1$; but the same general formulas apply to all cases of flow. There is, however, a particular limitation to the accuracy of the formulas as applied to the cases of axial and mixed flow.

In deriving the formulas it is assumed that r_1 and r_2 have the same values for all particles. This is strictly true in the Fourneyron type of turbine (Fig. 92), and approximately true in the Francis (Fig. 105). In the Jonval (Fig. 104) it is not true; but since the variation in the value of r for different particles is relatively small, it is fairly satisfactory in this case to use an average value as applying to all particles.

In the case of mixed flow represented in Fig. 106, the radius of admission r_1 is the same for all particles, but the range of values of r_2 for different particles is very great. By using a mean value of r_2 the formulas will still apply roughly. With any given wheel it will doubtless be approximately true that the highest efficiency results when the speed is such that the total angular momentum of the water leaving the runner is zero, but it does not seem possible to get a useful expression for the value of this angular momentum in terms of the constants and variables of the problem. The actual direction of motion of a particle leaving the runner at any point is uncertain even when the form of the runner is given, and theory cannot furnish definite rules for the design of the vanes.

227. Case of Discharge into Diverging Passages.—If the wheel discharges into passages which diverge so as gradually to reduce the velocity before the stream passes into the air or into a body of free water, the kinetic energy represented by the absolute velocity of outflow from the wheel may not be wholly lost.

Fig. 110 represents a "diffuser," devised by Boyden, used with a submerged wheel with outward flow. The vertical sec-

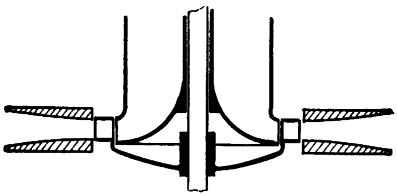


FIG. 110.

tion shows that the diffuser acts practically as a diverging tube, receiving the water from the wheel with absolute velocity V_2 and discharging it with a less velocity. Experiments by Francis * showed a gain of three per cent in the efficiency by the use of the diffuser.

To take account of the possible saving of energy due to discharge into diverging passages, the theory may be modified by replacing the term $V_2^2/2g$ in the value of h' by $k'(V_2^2/2g)$, k' being a fraction which may be assumed constant for the purpose of the theory. Carrying out the solution as before, the equation which replaces (A) is

$$(k+k')c'^2y^2+2[\cos A_1-(1-k')cc' \cos a_2]yx-(2-k')c^2x^2=1. \qquad (A')$$

No change is made in equation (B):

$$e=2(\cos A_1-cc' \cos a_2)yx-2c^2x^2. \quad . \quad . \quad . \quad (B')$$

The equation replacing (C), given by the condition $de/dx=0$, may be found without difficulty, but we will here use instead the equation resulting from the assumption $A_2=90°$, which is the same as above given:

$$cx+c' \cos a_2 \cdot y=0. \quad . \quad . \quad . \quad . \quad . \quad (C')$$

* Lowell Hydraulic Experiments, p. 5.

228. Effect of Suction Tube.—If the wheel discharges into a suction tube, the kinetic energy possessed by the water as it leaves the buckets may not be wholly lost. This will be true if the construction is such that the change of velocity of the stream in passing from the buckets into the tube is gradual. The energy corresponding to the velocity of outflow from the suction tube into the tail race will, however, be lost. It would seem that a saving might on the whole be effected by the use of a diverging tube, which should receive the water from the wheel with as little change of velocity as possible, and gradually reduce the velocity before discharging the stream into the tail race.

While it is not possible to express with any exactness the value of the loss of head in this case, the assumption made in Art. 227 will serve as a basis for an approximate solution of the problem of design for maximum efficiency. That is, it may be assumed that

$$h' = k\frac{v_2^2}{2g} + k'\frac{V_2^2}{2g},$$

in which k' lies between 0 and 1. If it be assumed also that the best speed is that which makes the tangential component of V_2 zero, the resulting equations are identical with those of Art. 227.

In this case there is a special reason for imposing the condition $A_2 = 90°$, for if this is not satisfied the water enters the suction tube with a whirling motion which must increase the loss of energy.

The pressure at any point within the suction tube depends upon the height above the tail water, the velocity of flow, and the loss of head in the tube. Thus let the equation of energy be written for the sections Y and Z (Fig. 104). If z, p, v refer to the point Y, p_0 = atmospheric pressure, and H' = loss of head between Y and Z,

$$H_1 = z + \frac{p}{w} + \frac{v^2}{2g},$$

$$H_2 = \frac{p_0}{w}.$$

$$\therefore \qquad z+\frac{p}{w}+\frac{v^2}{2g}-\frac{p_0}{w}=H',$$

or

$$\frac{p}{w}=\frac{p_0}{w}-z-\frac{v^2}{2g}+H'.$$

229. Regulation and Governing.—The problem of regulating the quantity of water supplied to the wheel, without serious loss of efficiency, is more difficult with reaction turbines than with those of the impulse type. With the latter it is only necessary to vary the cross-section of the guide passages at the place of discharge, and this can be done with relatively little loss of energy. With a reaction turbine, however, since the wheel-passages are always filled, a throttling of the stream at any point causes not only a contraction but a subsequent expansion of the stream, resulting in serious loss of energy. Perhaps the most common regulating device is a cylindrical gate fitting over the guide openings, which can be adjusted to any desired amount of opening. With such a gate a turbine can give its highest efficiency only when the gate is fully open.

Regulation by pivoted guide-vanes was introduced by Prof. James Thomson, with the inward-flow turbine designed by him. The same device has been used with modern American turbines of the Francis type. By simultaneous rotation of the pivoted vanes the area of the guide openings may be varied with little or no loss of energy. The angle A_1 also changes, but without important effect on the efficiency.

Governing also is accomplished in a satisfactory manner by means of the pivoted guides, these being actuated by some form of centrifugal governor.

230. Actual Efficiencies of Reaction Turbines.—The difficulty of making accurate efficiency tests is so great that comparatively few reliable data exist regarding actual efficiencies of turbines under practical working conditions. Such a test requires the accurate measurement of the quantity of water used per second, of the fall, and of the power output of the turbine. Laboratory tests upon full-sized machines are gener-

ally out of the question, and power plants in actual operation are rarely so arranged that the requisite measurements can be made with a good degree of accuracy. It must be remembered that in the foregoing theory it is the hydraulic efficiency that is referred to, and that this cannot be determined experimentally; it is necessary to measure the power obtained from the motor rather than that given up to it by the water. And in most cases this power output has to be measured electrically, thus introducing electrical losses of more or less uncertain value as well as errors due to the electrical measuring instruments.

The experiments of Francis at Lowell * upon an outward-flow (Fourneyron type) turbine under a fall of about 13 feet, gave an efficiency at full gate of 79 per cent. In this case the power output was measured by means of a friction brake, so that no electrical losses or uncertainties were involved. It is claimed that some of the best modern turbines give efficiencies as high as 87 per cent, and it is probable that hydraulic efficiencies as high as 85 per cent are often realized by well designed turbines when working with full gate opening. With ordinary methods of regulation the partial closing of the gate causes an important decrease in the efficiency. Regulation by pivoted guides, with the object of avoiding such decrease, has already been referred to (Art. 229).

Another important consideration affecting efficient working is the relation of wheel speed to fall. The above theory has made it clear that a turbine cannot work with equal efficiency under different falls unless the speed is varied in proportion to the square root of the fall. Under practical working conditions, however, the speed is nearly always required to maintain a constant value, while the fall is liable to fluctuations. This usually prevents the continuous realization of the highest efficiencies of which a turbine may be capable.

* Lowell Hydraulic Experiments.

EXAMPLES.

1. Let the dimensions of a reaction turbine be as follows: $A_1 = 28°$, $a_2 = 158°$, $r_1 = 3.375$ ft., $r_2 = 4.146$ ft., $F_1 = 6.537$ sq. ft., $f_2 = 7.687$ sq. ft. (these data are the same as those of Ex. 1, Art. 222). Suppose the discharge to take place through a diffuser whose outer radius exceeds that of the wheel by 3.5 ft., and whose vertical dimension diverges from .93 ft. to 1.68 ft. Neglecting losses of energy in the diffuser, and assuming $k = .25$, determine the best velocity and the highest efficiency.

[Notice that if losses of head in the diffuser are neglected, k' depends only upon the ratio of V_2 to the velocity of outflow from the diffuser. Calling the latter V_2', and assuming V_2 to be exactly radial ($A_2 = 90°$), the above dimensions give

$$\frac{V_2'}{V_2} = \frac{4.146 \times .93}{7.646 \times 1.68} = .300;$$

$$\frac{V_2'^2}{2g} = .09\frac{V_2^2}{2g};$$

hence $k' = .09$.] *Ans.* $x = .557$, $y = .868$, $e = .86$.

2. Solve the preceding example on the assumption that $k' = .5$.
Ans. $x = .548$, $y = .857$, $e = .83$.

3. The following data apply to a reaction turbine with axial flow, discharging into the air: $A_1 = 14°\ 40'$, $a_2 = 153°\ 50'$, $F_1/f_2 = .54$, mean radius $r = .5$ ft. Determine best velocity, highest efficiency, and proper value of vane angle a_1, assuming $k = .25$.

4. The following data are for a mixed-flow turbine with suction-tube: $A_1 = 15°$, $a_2 = 160°$, $F_1/f_2 = .75$, $r_1/r_2 = 1.5$, $r_1 = 2.5$ ft., $h = 30$ ft., quantity of water available 120 cu. ft. per sec. Assume $k = .25$, $k' = .9$. Determine best speed, highest efficiency, best value of a_1. Also estimate the values of F_1, f_1, f_2, and the power developed.

CHAPTER XX.

TURBINE PUMPS.

231. Relation of Pumps and Motors.—Turbine pumps are closely related to turbine motors of the reaction type. It is, in fact, quite possible to design a turbine which may operate either as motor or pump. The fundamental equations in the theory of turbine pumps are identical with those already given in the theory of reaction turbines, except that certain changes of notation are found advisable. It will be instructive, however, to develop the theory of pumps independently, introducing it by illustrations of an elementary character.

The most common forms of turbine pumps are known as *centrifugal pumps,* because of the fact that an important factor in their operation is the variation of pressure due to rotation (Art. 30).

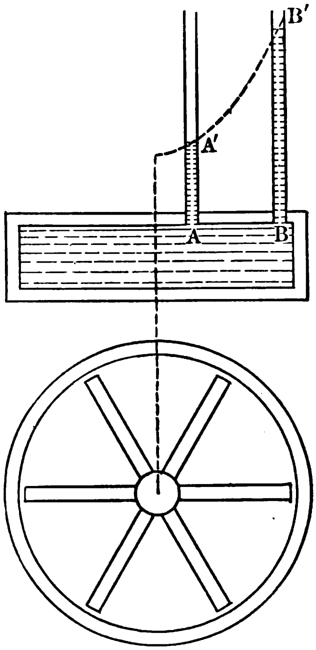


Fig. 111.

232. Column of Water Sustained by "Centrifugal Force."—Let a paddle-wheel rotate uniformly in a closed cylindrical chamber filled with water, the axis of rotation being vertical (see horizontal and vertical sections, Fig. 111). The water will tend to take up a condition of uniform rotation with the wheel. The whole body of water will not rotate as a rigid

body, because of friction against the surface of the chamber; but approximately such a condition of rotation will be reached. If vertical tubes communicate with the chamber at A and B, water will stand in them at unequal heights. The difference in level of the tops of the two columns may be computed from the results of Art. 30. If u_1 and u_2 are the velocities of the points A and B of the rotating body, the pressure head at B exceeds that at A by the amount $(u_2^2-u_1^2)/2g$; which is therefore also the vertical distance of B' above A'.

233. Crude Form of Centrifugal Pump.—Consider next a paddle-wheel rotating in a cylindrical chamber X (Fig. 112),

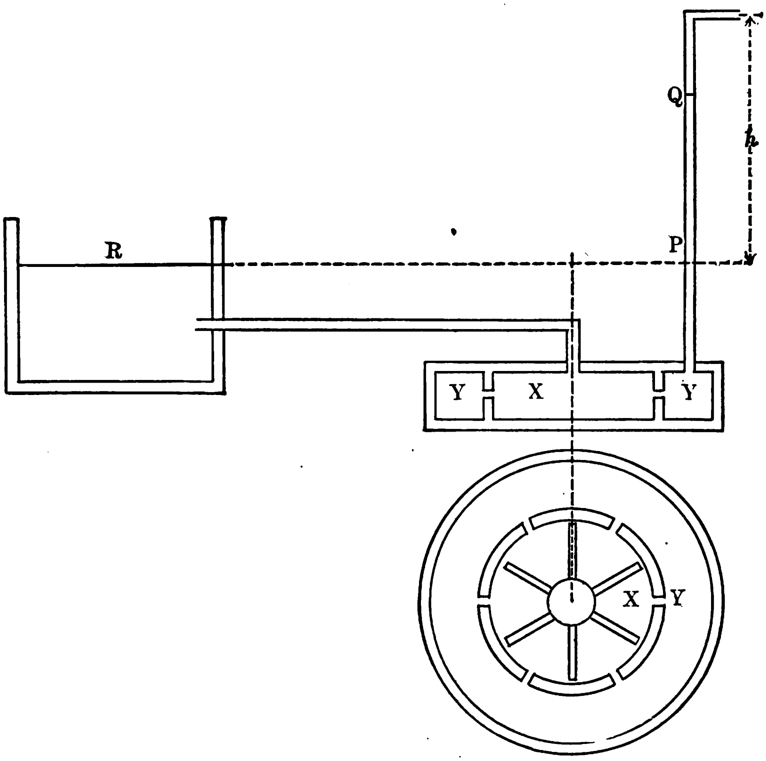


FIG. 112.

surrounded by a second chamber, Y, with which it communicates by means of orifices. A vertical pipe is connected with

the chamber Y, and another pipe leads from a point near the middle of the chamber X to a reservoir R. If the paddle-wheel is stationary, water fills both chambers and rises in the vertical pipe to a point P, level with the surface of the water in the reservoir. If the wheel rotates with angular velocity ω, the pressure in the chamber X will vary according to the law already given. If r is the radius of the inner chamber (practically equal to that of the wheel), the pressure at any point of the cylindrical bounding surface of the chamber X will exceed that at a point in the axis of rotation at the same level by the equivalent of a water column of height $r^2\omega^2/2g$. Pressure is communicated to the water in the outer chamber through the connecting orifices, and a condition of equilibrium will be reached in which the pressure within the chamber Y and the vertical pipe follows the hydrostatic law. At any point in this outer chamber the pressure will equal that at points on the same level at the outer boundary of the inner chamber. When equilibrium exists in the outer chamber the pressure conditions in the different parts of the apparatus are thus seen to be the following:

(*a*) Throughout the reservoir R and the connecting pipe, and at all points of the chamber X which lie in the axis of rotation, the pressure increases with the depth below the surface of the reservoir according to the hydrostatic law. (*b*) Throughout the chamber Y and the connecting pipe the pressure varies according to the hydrostatic law; but the pressure at any point exceeds that at points on the same level in the reservoir R by the equivalent of a water column of height $r^2\omega^2/2g$, or $u^2/2g$ (u being the linear velocity of the outer ends of the wheel-blades).

Evidently, therefore, water will rise in the pipe to a point Q, such that $PQ = u^2/2g$.

Let the open end of the pipe be at a height h above P. If the velocity of rotation of the wheel be such that

$$\frac{u^2}{2g} = h,$$

water will rise to the open end of the pipe; and if the velocity

of rotation exceeds this value, water will flow out. In this way a continuous stream can be caused to flow from the reservoir, discharging at the open end of the pipe; the possible height of the lift being proportional to the square of the wheel velocity. We thus have a centrifugal pump.

It is not necessary that the pump shall be lower than the surface of the supply reservoir; but if it is higher, some means must be provided for filling the chambers of the pump and the pipe leading from the reservoir, and to prevent the water from running back before the wheel velocity becomes great enough to sustain it. This may be accomplished by a foot-valve placed in the supply pipe, so arranged as to open only in the direction of flow. Neither is it necessary that the axis of rotation be vertical.

234. Transformations of Energy.—In such an apparatus as that shown in Fig. 112 energy must constantly be supplied to the wheel in order to maintain the uniform rotational velocity. This energy is given up to the water by the action of the wheel-blades. The particles of water next to the paddles, which receive energy directly from the wheel, pass it on to the neighboring particles, and through these it is transmitted to the particles remote from the moving blades. Since there will always be some relative motion of the particles of water among themselves, some energy is dissipated into heat by reason of internal friction in the water. Another portion of energy is dissipated by reason of the friction between the rotating body of water in the chamber X and the walls of the chamber. Such losses occur even if no flow takes place through the wheel.

If the wheel velocity is such that flow takes place, other losses of energy occur. There is evidently a frictional loss such as always accompanies the flow of water. Also, since a particle flowing through the apparatus suffers sudden changes of velocity, there is a loss due to this cause. Following a particle from the point where it enters the wheel chamber to a point in the discharge-pipe, it will be seen in a general way how its velocity varies. In moving from the axis of rotation to the circum-

ference of the chamber X, its velocity will be changed gradually from a small value to a value nearly equal to that of the outer ends of the paddles. As it enters one of the orifices leading into the outer chamber it is suddenly deflected; and on entering this chamber its velocity is nearly destroyed. These losses of energy are so important that such a crude pump as that represented in Fig. 112 would have a very low efficiency.

235. Design for Efficient Pump.—In Fig. 113* are shown the main features of a pump designed to reduce, as far as prac-

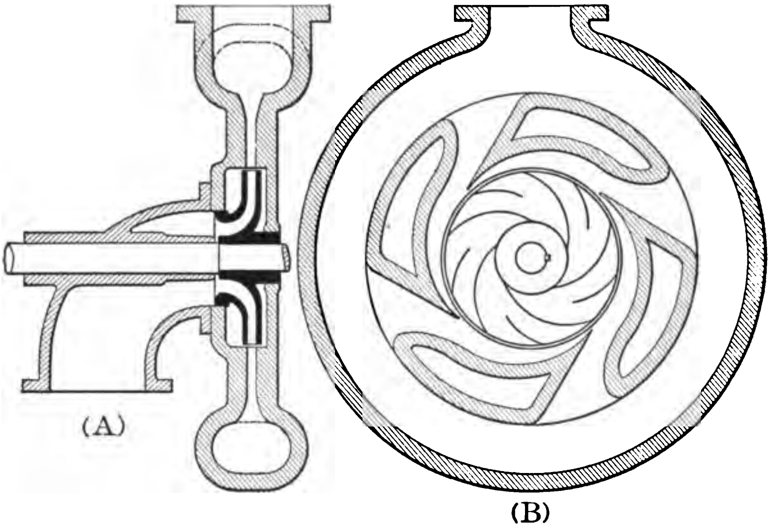


Fig. 113.

ticable, the losses of energy by dissipation into heat. The axis of rotation is horizontal,† (A) being a section by a plane containing this axis, and (B) a section by a plane perpendicular to the axis. The rotating wheel is formed with continuous passages, bounded by curved surfaces in such a way as to cause a gradual deflection of the water instead of a sudden deflection. The vanes which separate the passages are curved "backward," i.e., in such a way that the *relative* velocity of a particle of water when it reaches the outer circumference of the

* This figure shows the essential hydraulic features of the Risdon-Sulzer pump.

† The direction of the axis is not, however, essential.

wheel is directed as nearly as possible opposite to the velocity of the perimeter of the wheel at that point; the object being to make the *absolute* velocity of outflow as small as possible.

If it were possible to save all the kinetic energy possessed by the water as it leaves the wheel, the direction of the absolute velocity of outflow would be immaterial, so far as efficient working is concerned; it might be advantageous to make the angle a_2 as small as 90°, or even smaller, since by this means the rate at which the wheel would give energy to the water (when running at a given speed) would be increased. The reason for avoiding a high velocity of outflow from the wheel is that an important fraction of the kinetic energy due to this velocity is necessarily lost.

The absolute velocity of outflow must, however, have a forward tangential component, since the head-equivalent of the energy imparted by the wheel to the water is proportional to this component and to the speed of rotation.* It is important to conduct the water from the wheel to the receiving chamber in such a manner as to reduce the velocity of flow as gradually as possible. This is the aim of the design shown in Fig. 113. The water passes from the wheel into passages conforming to the direction of its absolute velocity just before leaving the wheel, and these passages enlarge very gradually to their place of discharge into the receiving chamber.

236. Relation between Lift and Wheel Speed. — It was shown above (Art. 233) that the height of the column of water that can be sustained by the rotation of the wheel *while no flow takes place* is $u_2^2/2g$, where u_2 is the velocity of the outer ends of the wheel-blades. When flow occurs, however, this relation no longer holds. Energy is continually given up by the wheel to the water, and this results in a gain of effective head which may take the form of an increased lift. If h''' is the head equivalent to the energy imparted to the water by the wheel (i.e., the energy given up by the wheel per pound of water discharged); h' the total loss of head by reason of dissipation of energy between supply reservoir and discharge

* This follows from formula (IV), as will be shown below.

reservoir; H_1 and H_2 the values of the effective head at surfaces of supply reservoir and discharge reservoir respectively; the general equation of energy gives (Art. 96)

$$H_2 - H_1 = h''' - h'.$$

The lift $H_2 - H_1$ is thus equal to the head-equivalent of the energy supplied by the wheel diminished by the frictional losses of head. If a wheel is so designed and operated as to impart energy to the stream efficiently (i.e., with small loss by dissipation), the lift maintained while flow occurs may be materially greater than that due to rotation when there is no flow. This is borne out by experience, some of the best pumps being found to give a lift greater than $u_2^2/2g$.

237. Notation for Mathematical Theory.—In the following mathematical theory the notation used in the theory of the reaction turbine will be employed, with the following modifications.

We now have to deal with a *lift* instead of a *fall*, and with *energy given up by wheel* instead of *energy received by wheel from water*. Hence we shall put

L = energy given up by wheel per second;

h = total lift from surface of supply reservoir to surface of discharge reservoir.

No change is made in the notation for absolute and relative velocities and their direction angles. It is, however, found convenient to express the equations in terms of u_2 and v_2, instead of u_1 and V_1 as in the former theory.

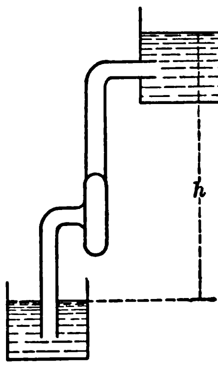


FIG. 114.

The discussion will refer to the case in which the admission and discharge occur substantially as in Fig. 113. The absolute velocity V_1 thus means the velocity of flow in the suction-pipe close to the wheel, and F_1 means the cross-section at this point. Since V_1 is parallel to the axis of rotation, $A_1 = 90°$ and $\cos A_1 = 0$. This greatly simplifies the main fundamental equations.

The direction of flow in the passages leading from the wheel to the receiving chamber depends upon the form of the passages. For the purpose of the mathematical theory it is necessary to idealize the conditions of flow within these passages by assuming that the direction angle of the velocity of a particle immediately after leaving the wheel has the same value for all particles. This angle, measured as usual from the direction of u_2, will be called A_2'. It must be remembered that A_2 is the direction angle of V_2, which is the absolute velocity of a particle *just before leaving the wheel.* For a given pump A_2 and A_2' can be equal only for some one ratio of wheel velocity to lift velocity. It is desirable that they should be equal when this ratio has the value under which the pump is designed to operate, since if they are unequal a sudden change of velocity occurs which results in loss of energy.

238. Fundamental Equations.—The energy given up by the wheel to the water is expended in two ways: in lifting the water a height h, and in overcoming the wasteful resistances represented by the lost head h'. Hence

$$L=W(h+h'). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The value of L may also be expressed by a formula similar to (IV) of Art. 179. In fact the reasoning leading to that formula (Art. 177) is strictly applicable to the case in which the energy given to the wheel by the water is negative. But with the present meaning of L we must change signs and write

$$L=\frac{W}{g}(u_2s_2-u_1s_1).$$

Or, since now $\qquad s_1=V_1 \cos A_1=0,$

we have $$L=\frac{W}{g}u_2s_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But $\qquad s_2=u_2+v_2 \cos a_2;$

hence
$$L=\frac{W}{g}u_2(u_2+v_2\cos a_2). \quad \ldots \ldots \quad (3)$$

Equations (1) and (3) now give

$$g(h+h')=u_2(u_2+v_2\cos a_2). \quad \ldots \quad (4)$$

Equations (3) and (4) are the fundamental equations upon which the following theory is based.

Equation (2) shows that if L has a positive value, s_2 must be positive; thus justifying the statement made in Art. 235 that the water must leave the wheel with a velocity having a forward tangential component.

239. Loss of Head.—To complete the theory it is necessary to estimate the value of h'. Doubtless the most important losses of head, in a pump working at its best speed for the existing lift, are of the kind included under the term friction losses. These depend upon the velocity of flow and the character of the passages, and may be expressed by means of a single term

$$k\frac{v_2^2}{2g},$$

in accordance with the usual assumption regarding such losses.

If there is a sudden change in the direction of the velocity of particles leaving the wheel, this causes a loss which cannot be expressed by a term of the above form. The passages into which discharge occurs should, if possible, be so formed that such sudden deflection of the water does not take place. This condition cannot be satisfied except for a particular relation between speed and lift, which should be that for which the pump is designed to work. In discussing the question of design for highest efficiency, it will be instructive to give first a solution of the problem assuming that no sudden deflection of the outflowing stream occurs.* Although the resulting equations will not show the working of an actual pump for different rela-

* This corresponds to the solution given in Art. 218 for turbine motors.

tions of speed to lift, they should agree well with actual conditions for the case which is to govern the design. The question of efficiency curves and discharge curves for a given pump will be considered afterward.

240. Design for Highest Efficiency.—Substituting in equation (4)

$$h' = k\frac{v_2^2}{2g},$$

we obtain

$$2u_2^2 + 2u_2v_2 \cos a_2 - kv_2^2 = 2gh, \quad . \quad . \quad . \quad . \quad (5)$$

from which v_2 and the rate of discharge can be computed for any wheel velocity.

The value of the efficiency may be expressed as follows: Since the energy utilized per second is Wh, while the energy supplied by the wheel to the water is $W(h+h')$, the hydraulic efficiency is

$$e = \frac{Wh}{W(h+h')} = \frac{h}{h+h'}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

or from (4),

$$e = \frac{gh}{u_2(u_2 + v_2 \cos a_2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The efficiency for any wheel speed is found by using in (7) the value of v_2 in terms of u_2 determined from equation (5).

The mathematical condition for maximum efficiency is $de/du_2 = 0$. The solution may be carried out by the method employed in Art. 220. In the present problem, however, the following method is shorter.

Since $h' = k\frac{v_2^2}{2g}$, the value of e in (6) may be written

$$e = \frac{2gh}{2gh + kv_2^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

This equation shows that, for any given value of h, e decreases continually as v_2 increases, having its greatest value when $v_2 = 0$. The maximum value of e is

$$e_m = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

while the corresponding value of u_2, found by putting $v_2=0$ in (5), is

$$u_2=\sqrt{gh}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The angle of outflow of the water from the wheel is

$$A_2=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

as may be seen from the relation between the three vectors representing u_2, v_2, V_2. Thus, since always

$$[V_2]=[u_2]+[v_2], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

the condition $v_2=0$ gives

$$[V_2]=[u_2]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The meaning of these results is more easily seen from a graphical representation.

Let $m=\sqrt{2gh}$, $x=u_2/m$, $y=v_2/m$. Equations (5) and (7) may then be written

$$2x^2+2xy\cos a_2-ky^2=1; \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$e=\frac{1}{2x(x+y\cos a_2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Taking x and y as rectangular coordinates, equation (14) represents an hyperbola whose center is at the origin. A particular case of this curve is represented at (A) in Fig. 115, which shows only the part of the curve for which y is positive. One asymptote of the hyperbola is shown in the figure.

Taking values of e as ordinates of a curve of which the abscissas are the corresponding values of x, the result is shown at (B), Fig. 115.

For a constant value of the lift the rate of discharge varies as y, and the speed as x. Hence curves (A) and (B) show how

the rate of discharge and the efficiency vary with the speed when the lift remains constant.

It must be remembered that these results do not apply to *a given pump* working at different speeds, but show the comparative results of an assumed series of pumps running at different speeds, assuming that in every case the absolute velocity

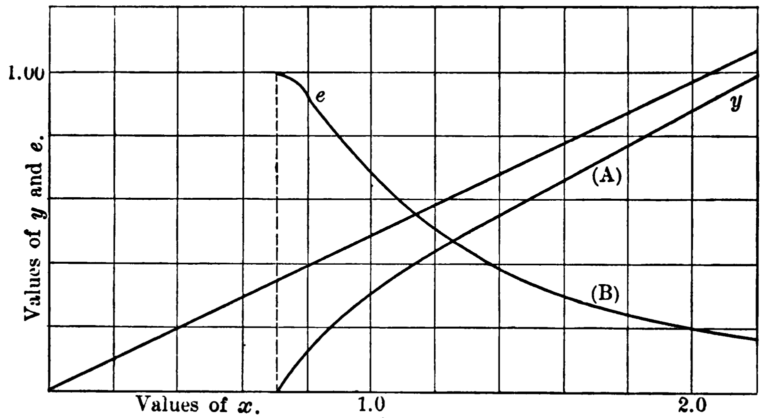


FIG. 115.

of outflow receives no sudden change, but that the flow from the wheel to the receiving chamber occurs with equal efficiency for all.

Strictly interpreted, the conclusion is that the case of maximum efficiency is that in which the angle A_2 is zero, the corresponding wheel velocity being $\sqrt{gh}$ or $.707m$, and the rate of discharge zero. As the ratio u_2/m or x increases, the efficiency decreases and the rate of discharge increases, the angle A_2 also increasing.

The practical conclusion is that the case of maximum efficiency is an ideal limit toward which the design should approximate, but which it can never reach, and that the speed of working should be such as to give a relatively small value of the discharge angle A_2. But in order to produce a given rate of discharge with as small a pump as practicable, it may be advisable to let the working speed increase materially above the value corresponding to the ideal case of maximum efficiency.

241. Losses of Head in a Given Pump.—In an actual pump working under different relations of speed to lift, the entire loss of head cannot be expressed by a term of the form assumed above. In addition to the "friction" losses there are losses due to sudden deflections of the stream as it enters and as it leaves the wheel, such deflections depending not upon the velocity of flow alone, but also upon the speed of rotation. Since the water enters the wheel near the axis of rotation where the wheel speed is relatively small, the deflection of the stream due to lack of exact adjustment of vanes to stream velocity at this point is probably of small importance. Sudden deflection at the point of outflow from the wheel would appear, however, to be an important cause of loss of head. While no exact evaluation of this loss can be made on theoretical grounds alone, the following considerations lead to an expression which probably represents actual conditions at least approximately for a pump discharging in the way shown in Fig. 113.

It may be noticed first that in the limiting case of no discharge ($v_2=0$) we ought to have $h'=u_2^2/2g$. For in this case the gain of effective head due to the action of the wheel (which is always the value of L/W given by (3)) reduces to u_2^2/g, while the actual lift has the value $u_2^2/2g$ by Art. 233, so that the lost head is the difference between these values. That $u_2^2/2g$ is a reasonable value for h' in this limiting case is seen also by considering what occurs when there is a very slight discharge. A particle just about to leave the wheel has absolute velocity u_2, while just outside the wheel it enters a body of water at rest,* so that its kinetic energy is wholly lost.

* Of course this is not an accurate statement of actual conditions, since the water in the passages just outside the wheel would be set in motion by the friction of the rotating body. With such a construction as that shown in Fig. 113, however, this water is prevented from taking up the motion of the wheel periphery, and the statement that the velocity of the particle leaving the wheel is wholly destroyed is practically true. The assumption above made as to the loss of head is in accordance with the usual rule for estimating loss of head due to a stream entering a body of water at rest. (Art. 82.)

Considering now the general case in which u_2 and v_2 have any values, we will make the ideal assumption referred to in Art. 237, that the direction angle of the absolute velocity of a particle of water just after leaving the rotating wheel has the same value for all particles, this value being fixed by the form of the passages leading to the receiving chamber. In other words, it will be assumed that the absolute velocity changes from a value V_2 with direction angle A_2 to a value V_2' with direction angle A_2', and that the latter angle has a fixed value, while the former varies with u_2 and v_2. In Fig. 116, AB, BC and AC represent u_2, v_2 and V_2 respectively; while BAC' is the fixed angle A_2', and V_2' is represented by some vector AC'. The absolute velocity thus changes from AC to AC' as the particle leaves the wheel.

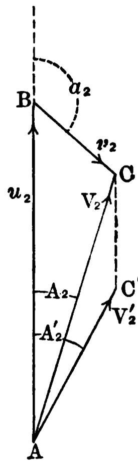


FIG. 116.

It will be assumed that the loss of head due to this sudden change is $(CC')^2/2g$.

The value of CC' is to be expressed in terms of u_2 and v_2. Since with proper construction (the thickness of the wheel-passages being equal to that of the passages into which the discharge occurs) the component of V_2 and that of V_2' perpendicular to u_2 are equal, CC' is parallel to AB, and its value is easily found to be

$$CC' = u_2 - \frac{\sin (a_2 - A_2')}{\sin A_2'} v_2 = u_2 - k'v_2, \quad . \quad . \quad . \quad (16)$$

if k' is written for the constant $\dfrac{\sin (a_2 - A_2')}{\sin A_2'}$.

The total loss of head may now be expressed by the equation

$$h' = k\frac{v_2^2}{2g} + \frac{(u_2 - k'v_2)^2}{2g}. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

242. Equations for Rate of Discharge and Efficiency.—Substituting the above value of h' in equation (4), the latter

takes the form

$$u_2^2+2(k'+\cos a_2)u_2v_2-(k+k'^2)v_2^2=2gh. \quad . \quad . \quad (18)$$

The value of the hydraulic efficiency in terms of u_2 and v_2, as given by equation (7), is not changed, but in applying it v_2 must have values satisfying (18). The equation is

$$e=\frac{gh}{u_2(u_2+v_2 \cos a_2)}. \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Introducing ratios of velocities, x and y having meanings as in Art. 240, the above equations become

$$x^2+2(k'+\cos a_2)xy-(k+k'^2)y^2=1; \quad . \quad . \quad . \quad (20)$$

$$e=\frac{1}{2x(x+y \cos a_2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

243. Condition of Maximum Efficiency.—The values of x and y for which e is a maximum may be found by applying the condition $de/dx=0$. Writing (21) in the form

$$\frac{1}{2e}=x^2+xy \cos a_2$$

and differentiating,

$$-\frac{1}{2e^2}\frac{de}{dx}=(2x+y \cos a_2)+x \cos a_2 \cdot \frac{dy}{dx}=0.$$

Differentiating (20),

$$[x+(k'+\cos a_2)y]+[(k'+\cos a_2)x-(k+k'^2)y]\frac{dy}{dx}=0.$$

Equating values of dy/dx given by these two equations, and reducing,

$$y^2+2xy \sec a_2-\frac{1+2k' \sec a_2}{k+k'^2}x^2=0. \quad . \quad . \quad . \quad (22)$$

This equation determines two values of y/x, one of which gives a real solution of the problem. Calling this value α, and putting $y=\alpha x$ in equation (20), the result is

$$[1+2(k'+\cos a_2)\alpha-(k+k'^2)\alpha^2]x^2=1, \quad . \quad . \quad . \quad (23)$$

which gives the value of x for maximum efficiency.

244. Graphical Representation for Case of Constant Lift.—When the lift remains constant, x varies directly as the wheel speed, and y as the rate of discharge. Hence the curve represented by (20), when x and y are made rectangular coordinates, shows how the rate of discharge varies when the wheel is run at different speeds under a constant lift.

The relation between efficiency and wheel speed under the same conditions may be represented by a curve of which the ordinate and abscissa of any point are corresponding values of e and x. The value of e for any value of x is to be computed from (20) and (21).

A third curve of importance is that showing the relation between wheel velocity and power. This relation is expressed by equation (3), or more conveniently by the equivalent equation

$$L=\frac{Wh}{e}.$$

Since W varies with varying speed, we substitute for it its value,

$$W=wq=wf_2v_2=wf_2my,$$

with the result

$$L=\frac{wf_2myh}{e}=\frac{wf_2m^3}{2g}\cdot\frac{y}{e}=\frac{wf_2m^3}{2g}l. \quad . \quad . \quad . \quad (24)$$

The variable factor in this value of L is y/e, which has been represented by l. If l be made ordinate of a point whose abscissa is x, the locus of this point as x varies is a curve showing the way in which the power varies with the wheel velocity.

The equations for the case of constant lift will for convenience be summarized below. With them will be included (22), which applies to the case of maximum efficiency, and which is seen to represent two straight lines, one of which intersects the general x-y curve in the point whose abscissa is the value of x for maximum efficiency.*

$$x^2+2(k'+\cos a_2)xy-(k+k'^2)y^2=1. \quad . \quad . \quad . \quad (A)$$

$$e=\frac{1}{2x(x+y\cos a_2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (B)$$

$$y^2+2xy\sec a_2-\frac{1+2k'\sec a_2}{k+k'^2}x^2=0. \quad . \quad . \quad (C)$$

$$q=f_2v_2=f_2my. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (D)$$

$$L=\frac{wf_2m^3}{2g}\cdot\frac{y}{e}=\frac{wf_2m^3}{2g}\cdot 2xy(x+y\cos a_2)=\frac{wf_2m^3}{2g}l. \quad . \quad (E)$$

Special case.—For a given wheel the angles a_2 and A_2' are known, and k' can be computed from them. The friction factor k cannot, however, be known apart from experiment. Consider a pump for which

$$a_2=150°, \quad A_2'=10°, \quad k'=\frac{\sin(a_2-A_2')}{\sin A_2'}=3.70.$$

Equations (A), (B), and (C) become

$$x^2+5.67xy-(13.70+k)y^2=1; \quad . \quad . \quad . \quad (25)$$

$$e=\frac{1}{2x(x-.866y)}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

$$y^2-2.31xy+\frac{7.55}{13.7+k}x^2=0. \quad . \quad . \quad . \quad . \quad (27)$$

* It will be noticed that there is a correspondence between these equations and those given in Art. 223 for the reaction turbine.

It appears that if k is small in comparison with 13.70, it has little influence on the form of the x-y curve. Fig. 117 shows the curves obtained by assuming* $k=0$. This reduces (25) and (27) to the forms

$$x^2+5.67xy-13.70y^2=1, \quad . \quad . \quad . \quad . \quad . \quad (28)$$

$$y^2-2.31xy+.552x^2=0, \quad . \quad . \quad . \quad . \quad . \quad (29)$$

while (26) is unchanged since it does not involve k explicitly.

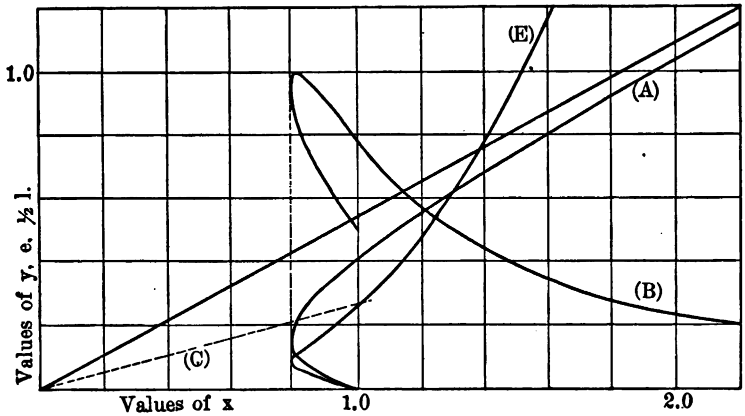


FIG. 117.

Equation (28) represents an hyperbola whose center is at the origin of coordinates. The curve cuts the x-axis in the point $x=1$, and the asymptotes are the lines

$$y=.547x, \quad y=-.1334x.$$

The coordinates of the vertex are $x=.795$, $y=.148$.

A remarkable feature of the curve is that for a certain range of values of x there are double values of y; i.e., to each value

* It is evident that this assumption makes the maximum efficiency 1, since h' will reduce to 0 when $A_2=A_2'$.

of the wheel velocity there correspond two values of the rate of discharge. Although this may at first sight seem anomalous, it is in harmony with the general explanation given in Art. 236. If the height of the column sustained by rotation without flow is $u_2^2/2g$ (in accordance with the theory of Art. 233), $y=0$ should give $x=1$. But when flow occurs, the lift may be greater than $u_2^2/2g$ because of the energy imparted by the rotating wheel to the water flowing through the wheel passages; that is, values of x less than 1 would give positive values of y. The soundness of this reasoning is corroborated by the results of experiments. Fig. 118 shows the x-y curve and the efficiency curve * obtained from experiments on a pump similar in design to the one considered in the foregoing theory.

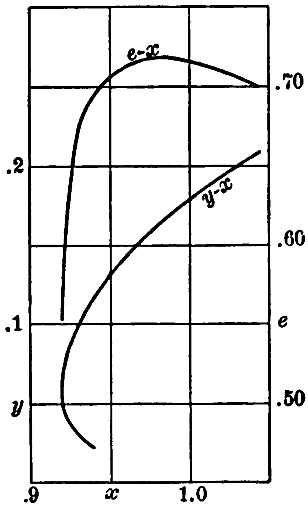


FIG. 118.

The curves in Fig. 117 are marked with the same letters as the corresponding general equations.

One solution of equation (29) is

$$y=.269x,$$

which represents a straight line intersecting the x–y hyperbola in the point whose coordinates are $x=.808$, $y=.218$, the values corresponding to maximum efficiency. The value of the maximum efficiency is 1, as it must be if $k=0$.

245. Effect of Friction Loss on Form of Curves.—In order to show how the form of the curves and the solution for maxi-

* It should be said that the efficiencies represented in the experimental curve are gross efficiencies computed from the actual work done in driving the pump, while the efficiencies given by equation (B) and represented by the corresponding curve in Fig. 117 are of course hydraulic efficiencies. The experimental curve is based upon data supplied by Mr. C. H. Stoddard, Chief Mechanical Engineer of the Risdon Iron Works, San Francisco.

mum efficiency are affected by the friction factor k, which has been assumed zero in the foregoing example, the curves for $k=5$ (all other data remaining unchanged) are shown in Fig. 119. Equations (25) and (27) now become

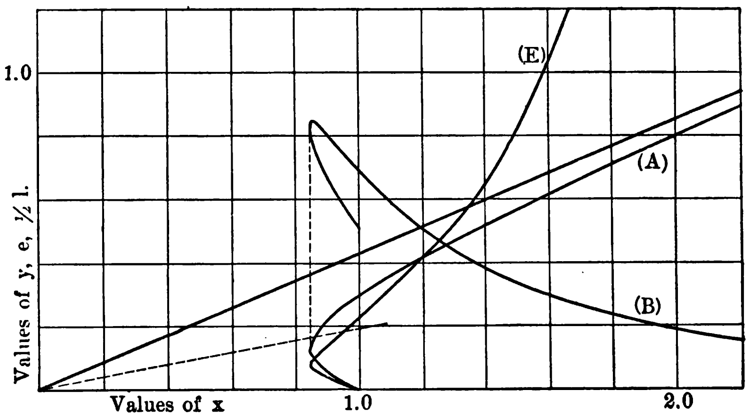


FIG. 119.

$$x^2+5.67xy-18.70y^2=1, \quad . \quad . \quad . \quad . \quad . \quad (30)$$

$$y^2-2.31xy+.403x^2=0. \quad . \quad . \quad . \quad . \quad . \quad (31)$$

The last equation gives for maximum efficiency $y=.189x$, which with (30) and (26) gives $x=.843$, $y=.160$, $e=.84$.

246. Remark on Friction Losses.—In the above theory h was defined as the total lift from supply reservoir to discharge reservoir, and h' as the total head lost between supply and discharge reservoirs. The reasoning is not changed, however, if $h=H_2-H_1$, where H_1 and H_2 are the values of the effective head at two sections A and B taken anywhere on the suction and discharge sides of the pump respectively, and h' is defined as the loss of head between these sections.* The value of v_2 for any

* That is, $H_1=z_1+p_1/w+V_1^2/2g$, $H_2=z_2+p_2/w+V_2^2/2g$. If H_1 and H_2 are to be determined by experiment, it is advantageous to have the cross-sections at A and B equal, so that the velocities need not be considered in determining H_2-H_1.

given value of u_2 will not be changed by changing the positions of the sections A and B, since the term $k(v_2^2/2g)$ expressing friction loss will increase or decrease exactly as H_2-H_1 decreases or increases. The values of x and y will, however, change, since these are ratios of u_2 and v_2 to h. The computed efficiencies will also depend upon the position of A and B, since the entire loss of head between these sections is charged against the pump when the efficiency is computed from equation (B).

EXAMPLES.

1. The following are dimensions of a centrifugal pump: $r_2=6.5''=.5417'$; $a_2=150°$; $f_2=.108$ sq. ft.; $A_2'=15°$. Determine x and y for maximum efficiency, neglecting friction losses.

Ans. $k'=2.732$, $x=.857$, $y=.314$, $e=1$.

2. Solve with same data except assume $k=6$.

Ans. $x=.903$, $y=.168$, $e=.73$.

3. With data as in Ex. 2, draw curves of discharge, efficiency and power under constant lift. *Ans.* $x^2+3.732xy-13.46y^2=1$.

4. With same data, if the lift is 150 ft., what number of R.P.M. should give highest efficiency, and what would be the corresponding rate of discharge? If the speed were increased so as to double the rate of discharge, what would be the efficiency?

Ans. For maximum efficiency $N=1565$ R.P.M., $q=1.785$ cu. ft. per sec.

$N=1870$ R.P.M. gives $q=3.570$ cu. ft. per sec., $e=.59$.

5. Solve examples 1 and 2 assuming $A_2'=30°$.

6. The following are dimensions of a pump: $r_2=8''$; $a_2=156°$; $f_2=5.25$ sq. in.; $A_2'=25°$. (*a*) Determine x and y for maximum efficiency if $k=4$. (*b*) Determine best speed of rotation and rate of discharge when the lift is 85 ft.

247. Curves for Rate of Discharge, Efficiency and Power when the Wheel Velocity Remains Constant.—An important practical question relates to the effect of variations of the lift when the pump runs at constant speed. The equations applying to such a case may be derived from those already given. It will be convenient to introduce the variable $m/u_2=1/x=x'$ instead of x, so that x' is proportional to $\sqrt{2gh}$ or m. Further, the rate of discharge is not proportional simply to y when the

lift varies, since m in equation (D) is not constant. Since $m=u_2/x$, (D) may be written

$$q=f_2u_2\frac{y}{x}.$$

We therefore introduce $y'=y/x$ as a new variable to represent the rate of discharge. The five main equations, expressed in terms of x' and y', become

$$x'^2+(k+k'^2)y'^2-2(k'+\cos a_2)y'=1; \quad . \quad . \quad . \quad \text{(A}'\text{)}$$

$$e=\frac{x'^2}{2(1+y'\cos a_2)}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(B}'\text{)}$$

$$y'^2+2y'\sec a_2-\frac{1+2k'\sec a_2}{k+k'^2}=0; \quad . \quad . \quad . \quad . \quad \text{(C}'\text{)}$$

$$q=f_2u_2y'; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(D}'\text{)}$$

$$L=\frac{wf_2u_2^3}{2g}\cdot 2y'(1+y'\cos a_2)=\frac{wf_2u_2^3}{2g}l'. \quad . \quad . \quad \text{(E}'\text{)}$$

In (E′), l' is written for the variable factor in the value of L; it is equal to l/x^3.

If curves are constructed with x' as abscissa and y', e, l', respectively, as ordinates, they will show the variation of rate of discharge, efficiency and power with the lift when the speed remains constant.

Special case.—For the particular data given above equations (A′) and (B′) become

$$x'^2+(13.70+k)y'^2-5.67y'=1, \quad . \quad . \quad . \quad . \quad (32)$$

$$e=\frac{x'^2}{2(1-.866y')}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Assuming as before $k=0$, the curves are shown at (A′), (B′), (E′), Fig. 120. The equation

$$x'^2+13.70y'^2-5.67y'=1$$

represents an ellipse whose principal axes are parallel to the axes of coordinates, and whose center is at the point $x' = 0$, $y' = .207$.

Of special interest is the form of curve (E′). This indicates that if the speed is constant, an increase in the lift causes a

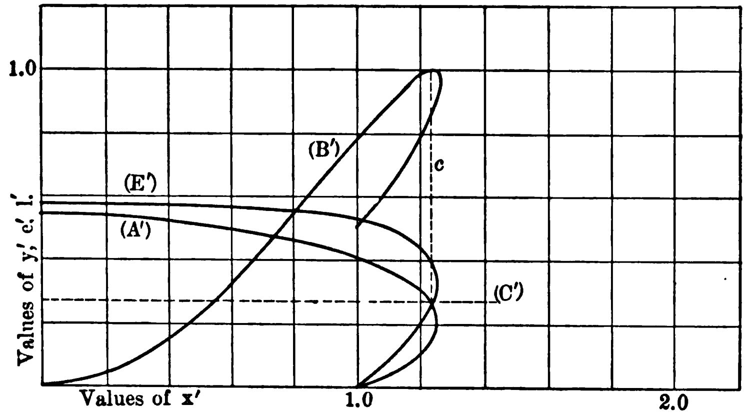


Fig. 120.

decrease in the power.* This explains what has sometimes been regarded as an anomaly in the practical working of centrifugal pumps. At first sight it might be inferred that an increased lift would increase the load upon the motor which drives the pump, but the opposite effect is often observed. This is in accordance with curve (E′), and is easily understood when it is considered that the power is proportional directly to the rate of discharge and to the lift, and inversely to the efficiency. An increase in the lift (the speed remaining constant) causes a decrease in the rate of discharge, and probably also an increase in the efficiency, so that the net result may be to decrease the power.

EXAMPLE.

With data as in Ex. 2, Art. 246, draw curves of discharge, efficiency and power for constant speed.

* Except for very small values of the rate of discharge.

248. Compounding.—A compound pump consists of two or more pumps arranged in series, i.e., so that the discharge of one is the supply of the next. Since the lift due to a wheel does not depend upon the actual value of the pressure upon it, the total lift due to such a series is the sum of the lifts which the several wheels would produce acting separately. For high lifts compounding is common, extremely high wheel velocities being thus avoided. The different wheels are commonly made alike in dimensions and mounted on the same shaft, so that they rotate together and produce equal lifts. See Fig. 121.

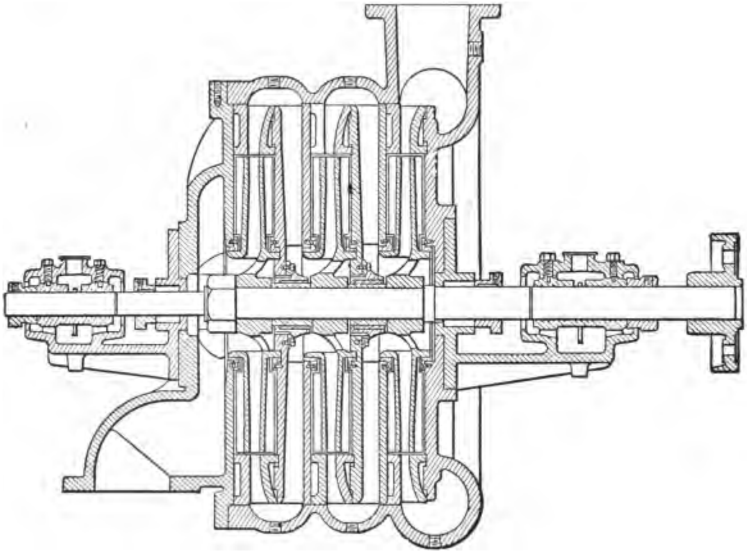


FIG. 121. Worthington three-stage turbine pump.

249. Balancing.—Since the impeller must have clearance space in order to rotate freely, it will be surrounded by water under pressure. Since friction will prevent this water from acquiring the full rotational velocity of the wheel, the intensity of pressure throughout the clearance space may approach the value existing at the periphery of the wheel. The intensity of pressure in the supply pipe being much less than this, the central portion of the wheel will experience a much greater total pressure from the back than from the front, thus subjecting the shaft to a thrust which must be provided for in the design.

In order to balance this thrust a common practice is to mount upon the same shaft two wheels alike in all respects except that they are right- and left-handed. Fig. 122 shows

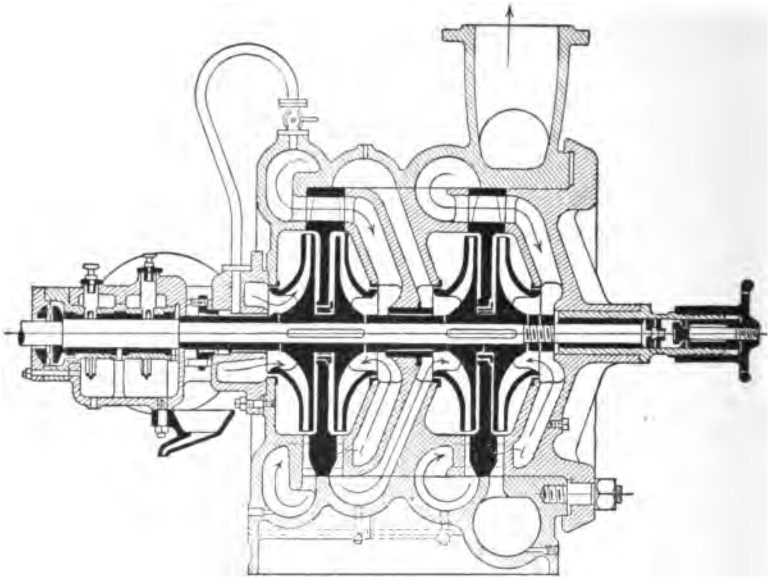


FIG. 122. Risdon-Sulzer four-stage pump with balanced impellers.

such a construction with two pairs of balanced runners, connected in series so as to form a four-stage pump.

250. Actual Lifts and Efficiencies.—It was formerly supposed that turbine pumps could be efficiently used only with small lifts and large discharges. Experiment has shown, however, that with proper design good efficiencies may be realized with very considerable lifts. Probably hydraulic efficiencies as high as 80 per cent can be obtained under a lift of more than 100 ft., with a single impeller wheel. By compounding, a lift of several hundred feet is entirely practicable.

APPENDIX A.

STEADY FLOW OF A GAS.

A 1. Effect of Compressibility on Theory of Steady Flow.—When the fluid is compressible the theory of steady flow requires important modification in two particulars. First, it is no longer necessary that equal volumes pass different sections in a given interval of time. Second, as the density of any definite portion of the fluid will in general change as it moves along the stream, it will do positive or negative work against the pressures acting on its bounding surface, resulting in the development of mechanical energy if the body expands, and the absorption of mechanical energy if it contracts.* Thus both the equation of continuity and the general equation of energy will be changed.

A 2. Equation of Continuity.—If the flow remains steady, equal *masses* of fluid must pass all cross-sections of the stream during any given time. But the mass passing a section F where the velocity is v and the density w is wFv per unit time. Hence, comparing different sections,

$$w_1F_1v_1 = w_2F_2v_2 = wFv = \text{constant.} \quad . \quad . \quad . \quad . \qquad (1)$$

A 3. Energy Passing a Given Cross-section.—The expression deduced in Art. 59 for the energy passing a given section of the stream is not changed by the fact that the fluid is compressible. But in applying it to different sections the appropriate value of w must be used at each section.

* In other words, there is a transformation of molecular energy into mechanical or the reverse.

A 4. Energy Generated within a Given Portion of the Stream by Expansion.—Let Fig. A 1 represent a portion of a steady stream of gas, and let it be assumed that the density decreases continuously in the direction of flow.* Let A and B be any two cross-sections, and consider the motion of the body of gas AB during a short time Δt, at the end of which it occupies the volume $A'B'$. If W denotes the mass of fluid which passes any section per unit time, we have for the mass passing a section during the time Δt

$$W\Delta t = w_1F_1v_1\Delta t = w_2F_2v_2\Delta t;$$

each of these expressions being equal to the mass of fluid in each of the volumes AA', BB'. During the small motion considered, each elementary portion of gas expands slightly, doing work against pressure; it is desired to determine the total amount of such work done during the time Δt by all the elementary portions of the body AB. Now in steady flow the condition at any given point of the volume $A'B$ must be the same at the end of the time Δt as at the beginning. The whole work of expansion done by the body AB is therefore equal to that which would be done by the elementary portion AA' if it expanded into the volume BB', passing through all stages as to temperature, pressure, and density that actually exist in the stream from A to B. This work cannot, therefore, be computed, unless the condition of the gas is known at every point in the stream AB.

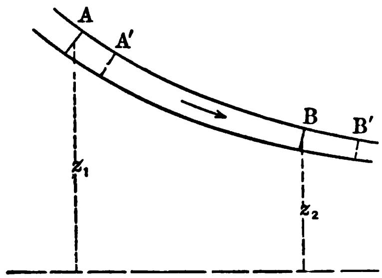


FIG. A 1.

Isothermal change of perfect gas.—If the temperature of the gas is the same at every point and remains unchanged during the flow, the relation between pressure and density for the case of isothermal expansion is to be used in computing the work.

* This is merely for convenience in stating the argument. The conclusion holds algebraically in any case.

For a perfect gas this relation is

$$\frac{p_1}{p_2}=\frac{w_1}{w_2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or

$$\frac{p}{w}=\frac{p_1}{w_1}=\frac{p_2}{w_2}=\text{constant}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The work done by unit mass in expanding from pressure p_1 and density w_1 to pressure p_2 and density w_2 is easily found to be

$$\frac{p_1}{w_1}\log\frac{p_1}{p_2}.$$

The work of expansion within the volume AB per unit time is therefore

$$W\frac{p_1}{w_1}\log\frac{p_1}{p_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Adiabatic change of perfect gas.—If no heat is given out or received by any portion of the gas during the flow, the condition of steady flow must be such that the temperature varies along the stream with the density in just the same way as it varies in a given portion of gas expanding or contracting adiabatically. For perfect gases the adiabatic law connecting pressure and density is

$$\frac{p_1}{p_2}=\left(\frac{w_1}{w_2}\right)^k, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

in which k is the ratio of the specific heat at constant pressure to that at constant volume, and has the value 1.41 very nearly. The work done by unit mass in expanding from pressure p_1 and density w_1 to pressure p_2 and density w_2 is easily shown to be

$$\frac{1}{k-1}\left(\frac{p_1}{w_1}-\frac{p_2}{w_2}\right).$$

Hence the work of expansion per unit time within the volume AB is

$$\frac{W}{k-1}\left(\frac{p_1}{w_1}-\frac{p_2}{w_2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Expansion of steam.—For saturated steam expanding adiabatically the result given for perfect gases holds approximately, except that a different number takes the place of k. Assuming the equation

$$\frac{p_1}{p_2}=\left(\frac{w_1}{w_2}\right)^m, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

the work of expansion within the volume AB per unit time is

$$\frac{W}{m-1}\left(\frac{p_1}{w_1}-\frac{p_2}{w_2}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

in which the value of m is about 1.13 for dry saturated steam and somewhat less for steam mixed with water. A value commonly used is $\frac{10}{9}$.

The cases of isothermal and adiabatic change are the two extremes between which any actual case of flow of a gas will lie. In order to maintain the isothermal condition there must be free conduction of heat between the stream and surrounding bodies, which are kept at constant temperature. Practically, the isothermal condition could be approximated to only in case of very slow flow. If the flow is very rapid, the quantity of heat gained or lost by conduction by any portion of the gas may be a small fraction of that absorbed or generated by reason of the work of expansion or contraction, so that the condition of adiabatic change may be nearly realized.

In any case, let WH'' represent the mechanical energy generated per unit time within the volume AB by reason of expansion. Then for isothermal flow

$$H''=\frac{p_1}{w_1}\log\frac{p_1}{p_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

For adiabatic flow

$$H''=\frac{1}{k-1}\left(\frac{p_1}{w_1}-\frac{p_2}{w_2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

A 5. General Equation of Energy for Steady Flow of a Gas. —Recurring now to the reasoning used in Art. 60, consider the gains and losses of mechanical energy in the volume AB (Fig. A 1) per unit time. The volume receives across the section A the amount of energy

$$W\left(z_1+\frac{p_1}{w_1}+\frac{v_1^2}{2g}\right).$$

It loses across the section B the amount

$$W\left(z_2+\frac{p_2}{w_2}+\frac{v_2^2}{2g}\right).$$

It loses by dissipation an amount which may be called WH'. And finally it gains an amount WH'' due to the work done by the gas in expanding. But the total energy gained must equal the total energy lost, since in steady flow the quantity of mechanical energy within the volume AB remains constant. Hence

$$z_1+\frac{p_1}{w_1}+\frac{v_1^2}{2g}+H''=z_2+\frac{p_2}{w_2}+\frac{v_2^2}{2g}+H', \quad . \quad . \quad . \quad (11)$$

or

$$H_1-H_2=H'-H'', \quad . \quad . \quad . \quad . \quad . \quad (12)$$

in which H_1 and H_2 have meanings as in Art. 62.

A 6. Equation of Energy for Isothermal Flow.—Using equation (9), and remembering that in isothermal change of volume $p_1/w_1=p_2/w_2$, the general equation of energy becomes

$$z_1+\frac{p_1}{w_1}(1+\log p_1)+\frac{v_1^2}{2g}=z_2+\frac{p_2}{w_2}(1+\log p_2)+\frac{v_2^2}{2g}+H'. \quad (13)$$

A 7. Equation of Energy for Adiabatic Flow.—In this case equation (10) is to be used, and the general equation of energy may be written in the form

$$z_1+\frac{k}{k-1}\cdot\frac{p_1}{w_1}+\frac{v_1^2}{2g}=z_2+\frac{k}{k-1}\cdot\frac{p_2}{w_2}+\frac{v_2^2}{2g}+H' \quad . \quad . \quad (14)$$

In applying this equation it must be remembered that p and w are connected by the relation (5).

Equation (14) applies approximately to steam, k being given the proper value.

A 8. Flow of Gas Through Small Orifice.—If a chamber (X, Fig. A 2) containing gas under pressure be connected by a small orifice or short tube with a second chamber (Y) in which a less pressure exists, flow will take place under practically adiabatic conditions. Neglecting loss of energy by dissipation, a formula for the velocity of flow through the orifice may be deduced from equation (14).

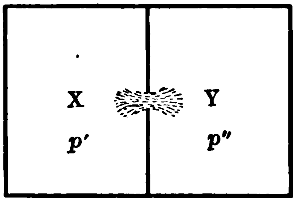


FIG. A 2.

Let p' be the pressure within the chamber X, p'' that within the chamber Y, and p_0 that within the stream passing through the orifice.* In applying the equation of energy, let the point corresponding to A, Fig. A 1, be taken within the chamber X, and the point corresponding to B at the smallest cross-section of the stream passing the orifice. The terms z_1 and z_2 may be neglected, and we have also

$$v_1=0, \quad p_1/w_1=p'/w', \quad v_2=v_0, \quad p_2/w_2=p_0/w_0,$$

so that the equation reduces to

$$\frac{v_0^2}{2g}=\frac{k}{k-1}\left(\frac{p'}{w'}-\frac{p_0}{w_0}\right). \quad \ldots \ldots \quad (15)$$

Or, making use of the relation between pressures and densities in adiabatic change, we may write

$$\frac{v_0^2}{2g}=\frac{k}{k-1}\,\frac{p'}{w'}\left[1-\left(\frac{p_0}{p'}\right)^{\frac{k-1}{k}}\right]. \quad \ldots \ldots \quad (16)$$

The velocity of flow through the orifice thus depends upon the ratio of the pressure within the orifice to that within the chamber X.

* It might seem that p_0 could be assumed equal to p'', but it will soon appear that this is not generally permissible.

Mass discharged per unit time.—If the cross-section of the jet is F_0, we have for the mass discharged per unit time

$$W = w_0 F_0 v_0 = \left(\frac{p_0}{p'}\right)^{\frac{1}{k}} w' F_0 v_0$$

$$= w' F_0 \sqrt{2g\frac{p'}{w'}\left(\frac{k}{k-1}\right)\left[\left(\frac{p_0}{p'}\right)^{\frac{2}{k}} - \left(\frac{p_0}{p'}\right)^{\frac{k+1}{k}}\right]}. \quad . \quad (17)$$

It will be seen that if p_0/p' be assumed to decrease from the value 1, W will increase up to a maximum and then decrease. The maximum occurs when

$$\frac{p_0}{p'} = \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

It appears, therefore, that p_0/p' will not fall below this value, however small the ratio p''/p'. Substituting this limiting value of p_0/p' in (16) and (17), the former becomes

$$\frac{v_0^2}{2g} = \frac{k}{k+1} \cdot \frac{p'}{w'}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and the latter takes the foım

$$W = w' F_0 \left(\frac{2}{k+1}\right)^{\frac{1}{k-1}} \sqrt{2g\frac{p'}{w'} \cdot \frac{k}{k+1}}. \quad . \quad . \quad . \quad (20)$$

The limiting values of p_0/p' for two cases, with the corresponding formulas for v_0 and W, are as follows:

Substance.	k	Limiting p_0/p'.	Formula for v_0.	Formula for W.
Air.	1.41	.527	$v_0 = .765\sqrt{2g\frac{p'}{w'}}$	$W = .485 w' F_0 \sqrt{2g\frac{p'}{w'}}$
Steam. . .	1.111	.582	$v_0 = .726\sqrt{2g\frac{p'}{w'}}$	$W = .446 w' F_0 \sqrt{2g\frac{p'}{w'}}$

The value of p'/w' for air at 0° C. is 26,200 ft., while at t° C. it is 26,200(1+.00366t). For any given temperature, therefore, the velocity as given by (19) is independent of the value of p'. For 0° C. it is $v_0=990$ ft. per sec.

For steam at an absolute pressure of 100 lbs. per sq. in. the value of p'/w' is about 63,300 ft.; at 200 lbs. per sq. in. it is about 66,200 ft. The corresponding values of the velocity of efflux as given by (19) are 1,460 ft. per sec. and 1,500 ft. per sec.

These results do not apply if p'' is greater than the value of p_0 given by (18); in that case equations (16) and (17) must be used, with $p_0=p''$.

A 9. Complete Solution of Problem of Adiabatic Flow.—In the case of flow considered above, in which $p''<p_0$, if steady flow continues beyond the section at which the pressure is p_0, it may be shown that both the cross-section and the velocity of the stream will increase as the pressure decreases. This is brought out in the following complete solution of the problem of adiabatic flow.

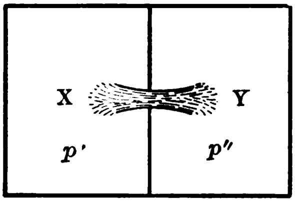


FIG. A 3.

Let p be the pressure and v the velocity at a point where the cross-section is F; then the relations between these three quantities are expressed by equations similar to (16) and (17):

$$\frac{v^2}{2g}=\frac{k}{k-1}\frac{p'}{w'}\left[1-\left(\frac{p}{p'}\right)^{\frac{k-1}{k}}\right]; \quad . \quad . \quad . \quad . \quad (21)$$

$$W=w'F\sqrt{2g\frac{p'}{w'}\left(\frac{k}{k-1}\right)\left[\left(\frac{p}{p'}\right)^{\frac{2}{k}}-\left(\frac{p}{p'}\right)^{\frac{k+1}{k}}\right]}. \quad . \quad (22)$$

By assuming values of p/p' the corresponding values of F and v can be computed from these equations. Or, using equa-

tions (19) and (20), the relations may be expressed in the following convenient forms:

$$\frac{v^2}{v_0{}^2}=\frac{k+1}{k-1}\left[1-\left(\frac{p}{p'}\right)^{\frac{k-1}{k}}\right]; \quad . \quad . \quad . \quad . \quad (23)$$

$$\frac{F_0}{F}=\left(\frac{k+1}{2}\right)^{\frac{1}{k-1}}\left(\frac{k+1}{k-1}\right)^{\frac{1}{2}}\left[\left(\frac{p}{p'}\right)^{\frac{2}{k}}-\left(\frac{p}{p'}\right)^{\frac{k+1}{k}}\right]^{\frac{1}{2}}$$

$$=\left(\frac{k+1}{2}\right)^{\frac{1}{k-1}}\left(\frac{p}{p'}\right)^{\frac{1}{k}}\frac{v}{v_0}. \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Application to air.—Putting $k=1.41$, these equations become

$$\frac{v^2}{v_0{}^2}=5.88\left[1-\left(\frac{p}{p'}\right)^{.291}\right]; \quad . \quad . \quad . \quad . \quad (25)$$

$$\frac{F_0}{F}=1.58\left(\frac{p}{p'}\right)^{.709}\frac{v}{v_0}. \quad . \quad . \quad . \quad . \quad . \quad (26)$$

The accompanying table gives values of v/v_0 and F/F_0 computed from these equations, together with the corresponding values of w/w'. The same results are shown graphically in Fig. A 4.

$\frac{p}{p'}$	$\frac{w}{w'}$	$\frac{v}{v_0}$	$\frac{F}{F_0}$	$\frac{p}{p'}$	$\frac{w}{w'}$	$\frac{v}{v_0}$	$\frac{F}{F_0}$
0	0	2.43	∞	.526	.634	1.00	1.00
.05	.119	1.85	2.87	.60	.696	.903	1.01
.10	.195	1.70	1.91	.70	.776	.762	1.07
.15	.260	1.58	1.54	.80	.853	.607	1.22
.20	.318	1.48	1.32	.85	.891	.521	1.37
.30	.426	1.32	1.13	.90	.928	.422	1.61
.40	.522	1.17	1.03	.95	.964	.296	2.22
.50	.614	1.04	1.01	1.00	1.000	0	∞

Application to steam.—The equations for steam, obtained by putting $k=\frac{10}{9}$, are as follows:

$$\frac{v^2}{v_0{}^2}=19\left[1-\left(\frac{p}{p'}\right)^{0.1}\right]; \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

$$\frac{F_0}{F}=1.627\left(\frac{p}{p'}\right)^{0.9}\frac{v}{v_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Values computed from these equations are given in the following table, and the corresponding curves are shown in Fig. A 4.

$\frac{p}{p'}$	$\frac{w}{w'}$	$\frac{v}{v_0}$	$\frac{F}{F_0}$	$\frac{p}{p'}$	$\frac{w}{w'}$	$\frac{v}{v_0}$	$\frac{F}{F_0}$
0	0	4.37	∞	.582	.614	1.00	1.00
.05	.067	2.22	4.10	.60	.631	.973	1.00
.10	.126	1.97	2.47	.70	.725	.817	1.04
.15	.181	1.81	1.87	.80	.818	.648	1.16
.20	.235	1.68	1.56	.85	.863	.555	1.28
.30	.338	1.47	1.24	.90	.909	.447	1.51
.40	.438	1.29	1.09	.95	.955	.311	2.07
.50	.536	1.13	1.02	1.00	1.000	0	∞

The above numerical results may be applied to any case of adiabatic flow of air or steam, if the pressure has known values p' and p'' at any two points. The two cases $p''<p_0$ and $p''>p_0$ require different treatment, as will be seen from the following examples.

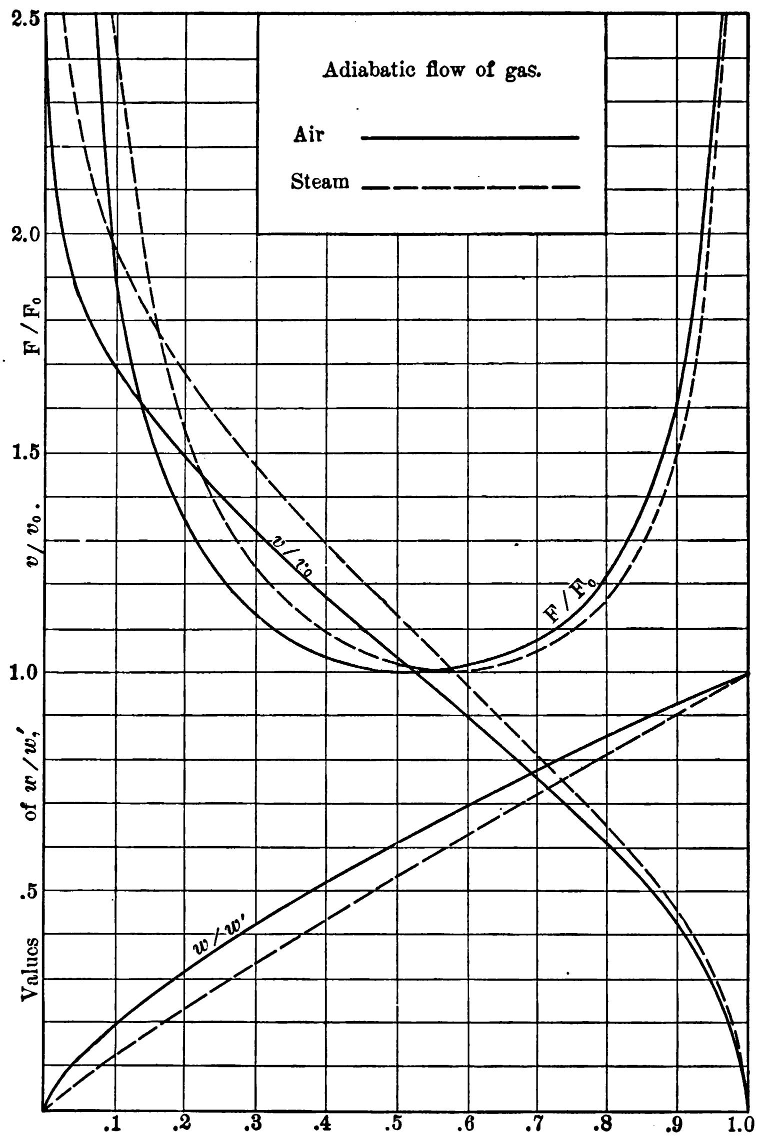
EXAMPLES.

1. A reservoir contains air at 20° C. under a pressure of 20 lbs. per sq. in. (absolute). Compute the mass discharged per second into the atmosphere through an orifice of one square inch area.

The rate of mass-discharge is $W=wFv$. In this case $p''>p_0$; we therefore take p'' as the pressure at the orifice, and find the corresponding values of w and v. For 20° C. we have $p'/w'=26200(1+.0732)=28100$ ft., and since $p'=2880$ lbs. per sq. ft., $w'=.1025$ lbs. per cu. ft. From the above table or diagram, $w/w'=.805$, hence $w=.0825$. Also $v/v_0=.705$; $v_0=.765\sqrt{64.4\times28100}=1030$; $v=.705\times1030=726$ ft. per sec. Finally, $W=wFv=.0825\times\frac{1}{144}\times726=.416$ lbs. per sec.

2. Air is discharged from a reservoir in which the pressure is 100 lbs. per sq. in. above atmospheric, and the temperature 20° C., through an orifice of one square inch area. Compute the mass discharged per second.

In this case $p''<p_0$, so that formulas (19) and (20) apply directly. The value of p'/w' is 28100 ft., as in Ex. 1; $p'=114.7\times144=16520$; $w'=16520/28100=.588$; $F_0=1/144$. Hence the formulas give $v_0=.765\sqrt{64.4\times28100}=1030$ ft. per sec.; $W=.485\times.588\times\frac{1}{144}\times\sqrt{64.4\times28100}=2.66$ lbs. per sec.



Values of p/p'

FIG. A4.

3. In Ex. 2, if the discharge is through a tube whose smallest cross-section is one square inch and which diverges beyond this section, determine the pressure and velocity at a section whose area is 1.2 square inches.

Here $F/F_0=1.2$, and Fig. A 4 gives $p/p'=.262$, $v/v_0=1.38$; hence $p=30.0$ lbs. per sq. in., $v=1420$ ft. per sec.

4. With conditions as in Ex. 3, what value of the cross-section of the tube would bring the pressure down to 14.7 lbs. per sq. in.? Determine the corresponding velocity.

Here $p/p'=14.7/114.7=.128$, $F/F_0=1.7$, $v/v_0=1.63$. Hence $F=1.7$ sq. in., $v=1680$ ft. per sec.

5. With conditions as in Ex. 3 except that the discharge is into a vacuum, what greatest velocity would be acquired?

Ans. If $p=0$, $v=2.43v_0=2500$ ft. per sec.

6. Steam under a pressure of 100 lbs. per sq. in. (above atmosphere) is discharged into the air through an orifice one square inch in area. Compute the mass discharged per second.

Since $p''<p_0$, formulas (19) and (20) apply with $k=10/9$. From steam tables we find $w'=.2580$ lbs. per cu. ft., hence $p'/w'=114.7\times144/.2580$ $=64100$ ft.; $v_0=1480$ ft. per sec.; $W=1.63$ lbs. per sec.

7. With data as in Ex. 6, if the discharge is through a diverging tube, compute the pressure and velocity at a point where the cross-section is 20 per cent greater than the smallest section.

Ans. $v=2110$ ft. per sec.; $p=36.7$ lbs. per sq. in. (absolute).

8. With conditions as in Ex. 7, what value of F/F_0 would bring the pressure down to 14.7 lbs. per sq. in.? Determine the corresponding velocity. *Ans.* $F/F_0=2.2$; $v=2780$ ft. per sec.

A 10. Flow of Gas in Pipe of Uniform Cross-Section.—The steady flow of a gas in a pipe of considerable length is in general neither isothermal nor adiabatic, but it seems reasonable to assume that these two cases are limits between which any actual case will lie. The following discussion indicates that the relation between pressure-gradient and rate of discharge is not greatly different in the two cases. It will be seen, in fact, that they are both covered by the same general theory, based upon the assumption that the relation between pressure and density is expressed by equation (5), the specific formulas for the different cases being obtained by substituting the proper values for the exponent k.

A 11. Frictional Loss of Head.—In order to apply the general equation of energy to the case of flow in a pipe it is necessary to express the value of the frictional loss of head. This is usually assumed to depend upon velocity, length and hydraulic radius in the same way as in the flow of water in a pipe (Arts. 80, 113), the formula for a pipe of circular cross-section being therefore

$$H' = f\frac{l}{d}\,\frac{v^2}{2g}\,. \qquad (29)$$

From a discussion of certain experiments Unwin * deduced the following empirical formulas for f, d being in feet:

For air $\qquad f = .0108\left(1 + \dfrac{3}{10d}\right)$.

For illuminating gas, $f = .0176\left(1 + \dfrac{1}{7d}\right)$.

It is probable that $f = .02$ is a safe value to use for gas in the absence of experimental knowledge for the particular gas in question.

A 12. Variation of Velocity Along the Pipe.—Since in steady flow equal masses of the fluid pass any two sections in equal times, it is evident that in a pipe of uniform cross-section the velocity varies inversely as the density, so that the velocity will be greatest where the pressure is least. Assuming the exponential relation (5) and combining it with (1), we have

$$\frac{v_1}{v_2} = \frac{w_2}{w_1} = \left(\frac{p_2}{p_1}\right)^{\frac{1}{k}}. \qquad (30)$$

A 13. Equation of Energy Applied to Flow in Pipe.—The fact that v varies along the pipe makes it necessary in applying (29) to consider a differential element of length. We therefore write the equation of energy (14) for an elementary length dx, the positive direction for x being taken as the direction of

* Treatise on Hydraulics, pp. 225, 230.

the flow. We also assume that $z_2 - z_1$ is negligible* in comparison with $\frac{p_1}{w_1} - \frac{p_2}{w_2}$. Equation (14) may then be written

$$\frac{k}{k-1} d\left(\frac{p}{w}\right) + d\left(\frac{v^2}{2g}\right) + \frac{f}{d} \cdot \frac{v^2}{2g} dx = 0. \quad . \quad . \quad (31)$$

At the section $x=0$ let $p=p_1$, $w=w_1$, $v=v_1$; then from (1) and (5),

$$\frac{v}{v_1} = \frac{w_1}{w} = \left(\frac{p_1}{p}\right)^{\frac{1}{k}}, \quad . \quad . \quad . \quad . \quad . \quad (32)$$

$$\frac{p}{w} = \frac{p_1}{w_1}\left(\frac{p}{p_1}\right)^{\frac{k-1}{k}}, \quad . \quad . \quad . \quad . \quad . \quad (33)$$

$$\frac{v^2}{2g} = \frac{v_1^2}{2g}\left(\frac{p_1}{p}\right)^{\frac{2}{k}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

From these relations (31) may be reduced to a differential equation between the two variables p and x. Thus, dividing through by $(v^2/2g)dx$, and substituting for $\frac{d}{dx}\left(\frac{p}{w}\right)$ and $\frac{d}{dx}\left(\frac{v^2}{2g}\right)$ their values obtained from (33) and (34), we obtain

$$\frac{2g}{w_1 v_1^2}\left(\frac{p}{p_1}\right)^{\frac{1}{k}} \frac{dp}{dx} - \frac{2}{kp}\frac{dp}{dx} + \frac{f}{d} = 0. \quad . \quad . \quad . \quad (35)$$

Integrating, and expressing the constant of integration in terms of p_1,

$$\frac{2gp_1}{v_1^2 w_1} \cdot \frac{k}{k+1}\left[1 - \left(\frac{p}{p_1}\right)^{\frac{k+1}{k}}\right] + \frac{2}{k}\log\frac{p}{p_1} = \frac{fx}{d}. \quad . \quad . \quad (36)$$

From this equation, if the pressure, density and velocity at one section are known, and if the value of f is known, the pressure at any section may be computed.

If reasoning similar to the above be applied, except that equation (13) is used instead of (14), there results an equation identical with (36) with $k=1$. Unless p/p_1 becomes less than any value likely to occur in practice, the difference between the two cases $k=1$, $k=1.4$ is unimportant. Values of

* If the pressure is so nearly uniform that this assumption is not allowable, the variation of w is so small that the fluid may be treated as incompressible and the ordinary formula of Hydraulics applied.

$\frac{k}{k+1}\left[1-\left(\frac{p}{p_1}\right)^{\frac{k+1}{k}}\right]$ for the two cases are shown in the accompanying table and in Fig. A 5. The curves show at a glance the degree of importance of the difference between the two

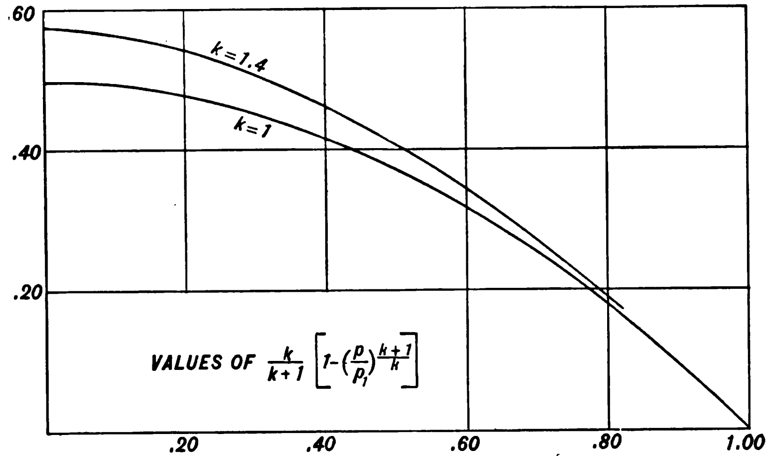


FIG. A 5.

extreme assumptions of isothermal and adiabatic flow,* the term $\frac{2}{k}\log\frac{p}{p_1}$ being negligible unless p/p_1 is smaller than is likely to be the case in practice.

$\frac{p}{p_1}$	$\frac{k}{k+1}\left[1-\left(\frac{p}{p_1}\right)^{\frac{k+1}{k}}\right]$		
	$k=1$	$k=\frac{4}{3}$	$k=1.4$
0	.500	.5714	.5833
.1	.495	.5613	.5721
.2	.480	.5373	.5464
.3	.455	.5019	.5092
.4	.420	.4564	.4620
.5	.375	.4015	.4055
.6	.320	.3377	.3403
.7	.255	.2653	.2668
.8	.180	.1847	.1854
.9	.095	.0963	.0964
1.0	0	0	0

* Unwin expresses the opinion that the proper assumption is that of isothermal flow, because the energy dissipated by friction and eddies is transformed into heat, which compensates for the effect of expansion in lowering the temperature. (Treatise on Hydraulics, p. 221.)

A 14. Formula for Mass-Rate of Discharge.—Let W lbs. per sec. be the rate of discharge for a pipe of diameter d ft. when the absolute pressure has values p_1, p_2 (lbs. per sq. ft.) at sections l ft. apart. Then

$$W = \tfrac{1}{4}\pi d^2 w_1 v_1,$$

in which v_1 may be computed from (36) with $x=l$, $p=p_2$. Assuming $k=1$ and neglecting the unimportant term, we have

$$v_1 = \sqrt{\frac{gp_1 d}{w_1 f l}\left(\frac{p_1^2 - p_2^2}{p_1^2}\right)}, \quad . \quad . \quad . \quad . \quad . \quad (37)$$

$$W = \frac{\pi}{4}\sqrt{\frac{gw_1 d^5 (p_1^2 - p_2^2)}{p_1 f l}} = \frac{\pi}{4}\sqrt{\frac{gd^5 (p_1^2 - p_2^2)}{RT_1 f l}}, \quad . \quad (38)$$

in which T_1 is the absolute temperature of the gas when the pressure is p_1 and density w_1, and R is a constant. Here we have made use of the relation

$$\frac{p}{wT} = R = \text{constant}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

which holds for a perfect gas.

A 15. Volume-Rate of Discharge.—Gas sold for domestic use is measured by volume at a standard pressure. Let p_0 be this standard pressure, and let q_0 denote the equivalent of W expressed in cubic feet per second at pressure p_0 and temperature T_0, the corresponding density being by (39)

$$w_0 = \frac{p_0}{RT_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

Then

$$q_0 = \frac{W}{w_0} = \frac{WRT_0}{p_0} = C\sqrt{\frac{d^5(p_1^2 - p_2^2)}{l}}, \quad . \quad . \quad (41)$$

in which

$$C = \frac{\pi T_0}{4p_0}\sqrt{\frac{gR}{T_1 f}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

A 16. Values of Constants.—The numerical value of the constant R for a given gas depends upon the units of length and temperature employed. Data for air and for two speci-

mens of gas are given in the accompanying table. There are also given values of $C\sqrt{f}$ based upon the assumptions $p_0 =$ 2120 lbs. per sq. ft., $T_0 = 520°$ (equivalent to about 60° F. on the ordinary scale), $T_1 = T_0$. If the Centigrade scale is used the values of R must be multiplied by 1.8.

	$p = 2120$ lbs. per sq. ft.				R	$C\sqrt{f}$
	$T = 492°$ F.		$T = 520°$ F.			
	w lbs. per cu. ft.	$\frac{p}{w}$ ft.	w lbs. per cu. ft.	$\frac{p}{w}$ ft.		
Air	.0807	26260	.07636	27760	53.4	.350
Natural gas	.0476	44530	.04504	47070	90.5	.456
Illuminating gas	.0324	65430	.03066	69160	132.9	.553

EXAMPLES.

1. Estimate the rate of discharge of air in a 6-inch pipe, the gauge pressures at two sections 2 miles apart being 80 lbs. and 50 lbs., assuming a uniform temperature of 60° F.

Ans. Taking $f = .0173$ (Unwin's formula), we find $C = 2.662$, $W = 3.482$, $v_1 = 36.10$, $v_2 = 52.83$. If $T_0 = T_1 = 520$, $q_0 = 45.60$.

2. Illuminating gas whose density at a pressure of 2120 lbs. per sq. ft. and temperature 32° F. is .0324 lb. per cu. ft. flows in a 12-inch pipe. The gauge pressures at two sections 4000 ft. apart are 40 lbs. and 30 lbs. Assuming a uniform temperature of 60° F. and $f = .02$, determine the rate of discharge in lbs. per sec., also in cu. ft. per sec. at 60° F. and zero gauge pressure (atmospheric pressure being 2120 lbs. per sq. ft.).

Ans. $C = 3.91$, $W = 8.60$, $q_0 = 281$, $v_1 = 96.1$, $v_2 = 117.6$.

3. The natural gas described in the above table is to be delivered at a certain point at the rate of 300,000 cu. ft. per hr. (estimated at 60° F. and absolute pressure 2120 lbs. per sq. ft.), and at a gauge pressure of 5 lbs. per sq. in. If the diameter of the pipe is 6 inches, what gauge pressures must be maintained at points 1000 ft., 2000 ft. and 3000 ft. from the place of delivery? Assume that the elevation above sea level is 4000 ft., and estimate atmospheric pressure as in Art. 14, Table II. Take $f = .02$, and assume $T_1 = T_0 = 520$.

Ans. Absolute pressures in lbs. per sq. ft. are 3455, 5193, 6580. Gauge pressures in lbs. per sq. in. are 24.0, 36.1, 45.7. (Atmospheric pressure is taken as 1820 lbs. per sq. ft.)

APPENDIX B.

RELATIVE MOTION.

B1. Meaning of Relative Motion.—In the theory of turbines we are concerned with the motion of a particle of water with respect to the earth, and also with its motion with respect to the rotating wheel. In order to understand just what is meant by these two motions, consider the following illustration.

Conceive a horizontal platform to be rotating uniformly (with respect to the earth) about a fixed vertical axis, and suppose an observer stationed upon the platform, to whom the earth is invisible and inaccessible. He will regard the platform as at rest, no other body being available with which he can compare its motion. A body moving on the platform would trace upon it a certain line which to the observer would be its true path. At any instant it would appear to be moving in this path in a certain direction at a certain rate, and to the observer this would be its true velocity. If the body were visible to a second observer stationed upon the earth, it would to him appear to follow quite a different path, and at any instant it would appear to be moving at a different rate and in a different direction.

By the *motion relative to the platform* would be meant the motion as estimated by the observer on the platform, while the *motion relative to the earth* would mean the motion as estimated by the observer on the earth.

Similarly, in case of water flowing through a rotating wheel, the motion of a particle relative to the wheel means its motion as it would be estimated by a person to whom the wheel appeared stationary.

B 2. Absolute and Relative Motion.—In the theory of turbines, as in all ordinary practical problems, the earth is regarded as a fixed body, and motion with respect to it is for convenience called "absolute," while motion with respect to the rotating wheel is called "relative."

It is necessary to consider definitely the relation between these two motions.

B 3. Relation between Absolute and Relative Velocities of a Particle.—Let the wheel rotate with uniform angular velocity ω about an axis represented in Fig. B 1 by the point O, the axis

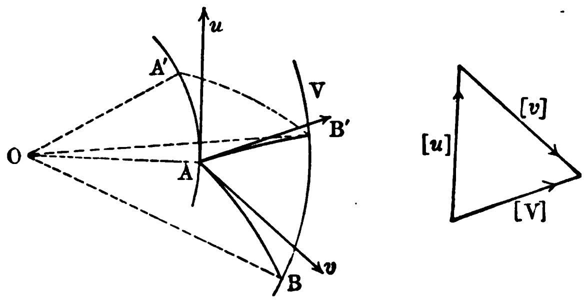


FIG. B 1.

being perpendicular to the plane of the figure. Let AB be the path traced by a particle upon the rotating wheel during a time Δt. Every point of the wheel describes a circle about the axis O; the point initially at A describes during Δt an arc AA', and the point initially at B an arc BB', the angles AOA', BOB' being equal. The moving particle describes with respect to the earth a path from A to B'.

Let V denote the absolute velocity of the particle when at A, v its relative velocity when at A, and u the absolute velocity of the point A of the wheel. These velocities are directed along the tangents to the curves AB', AB, AA', respectively, at the point A. Their vector values are *

$$[V] = \text{limit}_{(\Delta t - 0)} \frac{\text{vector } AB'}{\Delta t};$$

* Theoretical Mechanics, Art. 244. The brackets are used to denote vector values, as explained in Art. 174.

$$[v] = \text{limit}_{(\Delta t-0)} \frac{\text{vector } AB}{\Delta t};$$

$$[u] = \text{limit}_{(\Delta t-0)} \frac{\text{vector } AA'}{\Delta t}.$$

In the limit the figure $ABB'A'$ is a parallelogram, and

$$\text{vector } AB' = \text{vector } AB + \text{vector } AA',$$

hence $$[V] = [v] + [u].$$

That is, in words,

The absolute velocity of a particle is equal to the vector sum of its relative velocity and the absolute velocity of that point of the wheel which momentarily coincides with the particle.

Notice that the reasoning holds even if AB and AB' are not in a plane perpendicular to the axis, so that they are not shown in true size in the figure.

B 4. Relation between Absolute and Relative Accelerations.—Space will not be taken to deduce the relation between absolute and relative accelerations, since it is not required in the theory of turbines as given in the text. It is discussed in the author's Theoretical Mechanics, Chapter XXIV.

EXAMPLES.

1. A wheel rotates uniformly at the rate of 120 R.P.M. A particle 4 ft. from the axis has a velocity relative to the wheel of 30 ft. per sec. directed at an angle of 45° to the perpendicular from the particle to the axis of rotation. Compute its velocity relative to the earth.

Ans. 74.6 ft. per sec. directed at angle 73° 28′ to the radial line.

2. A wheel rotates uniformly at the rate of 120 R.P.M. A particle 4 ft. from the axis has an absolute velocity of 30 ft. per sec. directed at an angle of 45° to the perpendicular from the particle to the axis of rotation. Compute its relative velocity.

Ans. 36.0 ft. per sec. directed at angle 53° 50′ to the radial line.

APPENDIX C.

CONVERSION FACTORS.

THE following table, designed to facilitate the reduction from one unit or system of units to another, includes only such factors as are likely to be of use in the problems of practical Hydraulics. Five-figure values are in most cases given, although hydraulic measurements do not often warrant this degree of accuracy.

The numbers which involve the density of water are based upon the value for pure water at maximum density. The variation in density by reason of impurities (except in the case of sea-water or water in salt lakes) is ordinarily inappreciable, and the correction for temperature can be applied by aid of the table in Art. 8.

The ratio of the density of mercury to that of water has been taken as 13.596.

LENGTH.

				Log.
1 inch	=	2.5400	centimeters	.40483
1 foot	=	30.479	centimeters	1.48401
1 mile	=	1.6093	kilometers	.20664
1 centimeter	=	0.39371	inch	$\bar{1}$.59517
	=	0.032809	foot	$\bar{2}$.51599
1 kilometer	=	0.62138	mile	$\bar{1}$.79336

AREA.

1 square inch	=	6.45137	square centimeters	.80965
1 square foot	=	928.997	square centimeters	2.96801
1 square centimeter	=	0.15501	square inch	$\bar{1}$.19035
1 square meter	=	10.76430	square feet	1.03199

VOLUME.

				Log.
1 cubic inch	=	16.386	cubic centimeters	1.21448
1 cubic foot	=	0.028315	cubic meter	$\bar{2}$.45202
	=	28.315	liters	1.45202
	=	7.4805	U. S. gallons	.87393
	=	6.2321	British Imperial gallons	.79463
1 cubic centimeter	=	0.061027	cubic inch	$\bar{2}$.78552
1 liter (cubic decimeter)	=	61.027	cubic inches	1.78552
	=	0.035317	cubic foot	$\bar{2}$.54798
	=	0.26419	U. S. gallons	$\bar{1}$.42191
1 cubic meter	=	35.317	cubic feet	1.54798
1 U. S. gallon	=	231.	cubic inches	2.36361
	=	0.13368	cubic foot	$\bar{1}$.12607
	=	3.78521	liters	.57809
	=	0.83311	British Imperial gallon	$\bar{1}$.92070
1 British Imperial gallon	=	277.27	cubic inches	2.44291
	=	0.16046	cubic foot	$\bar{1}$.20537
	=	4.5435	liters	.65739
	=	1.2003	U. S. gallons	.07930

MASS AND WEIGHT.

1 pound	=	0.45359	kilogram	$\bar{1}$.65667
1 kilogram	=	2.2046	pounds	.34333

WEIGHT OF WATER (AT 4° CENT.).

1 cubic foot weighs	62.424	pounds	1.79535
	28.315	kilograms	1.45202
1 U. S. gallon weighs	8.3448	pounds	.92142
	3.7852	kilograms	.57810
1 Brit. Imp. gal. weighs	10.0165	pounds	1.00072
	4.5435	kilograms	0.65739
1 cubic meter weighs	1000.	kilograms	3.00000
	2204.7	pounds	4.34333

VELOCITY.

1 foot per second	=	0.68182	miles per hour	$\bar{1}$.83367
1 mile per hour	=	1.4667	feet per second	0.16633

RATE OF DISCHARGE.

1 cubic foot per second	=	7.4805	U. S. gallons per second	.87393
	=	448.83	U. S. gallons per minute	2.65208
	=	6.2321	B. I. gallons per second	.79463
	=	373.93	B. I. gallons per minute	2.57278
1 U. S. gallon per second	=	0.13368	cubic foot per second	$\bar{1}$.12607

RATE OF DISCHARGE—*Continued.*

				Log.
1 U. S. gallon per minute	=	0.0022280	cubic foot per second	$\bar{3}$.34792
1 million U. S. gallons per day	=	1.5472	cubic feet per second	.18956
1 B. I. gallon per second	=	0.16046	cubic foot per second	$\bar{1}$.20537
1 B. I. gallon per minute	=	0.0026743	cubic foot per second	$\bar{3}$.42722
1 million B. I. gallons per day	=	1.8572	cubic feet per second	.26886

PRESSURE.

1 pound per square inch	=	144.	pounds per square foot	2.15836
	=	70.310	grams per sq. centimeter	1.84702
	=	2.3068	feet water column	.36301
	=	2.0360	inches mercury column	.30878
1 pound per square foot	=	4.8826	kilograms per sq. meter	.68866
	=	0.016020	feet water column	$\bar{2}$.20465
	=	0.014139	inches mercury column	$\bar{2}$.15042
1 kilogram per sq. meter	=	0.20480	pounds per square foot	$\bar{1}$.31134
	=	0.1	centimeter water column	$\bar{1}$.00000
	=	0.0073552	centimeter mercury column	$\bar{3}$.86659
1 atmosphere	=	30.00	inches mercury column	1.4771
	=	34.00	feet water column	1.5315
	=	2122.	pounds per square foot	3.3268
	=	14.73	pounds per square inch	1.1685
	=	76.00	centimeters mercury column	1.8808
	=	10.33	meters water column	1.0142
	=	10330.	kilograms per sq. meter	4.0142
	=	1033.	grams per sq. centimeter	3.0142

INDEX.

ENGINEERING BOOKS

Hoskins's Hydraulics.

By L. M. HOSKINS, Professor in Leland Stanford University. 8vo. 271 pp. $2.50.

A comprehensive text-book, intended for the fundamental course in the subject usually offered in schools of engineering, but somewhat more compact in treatment than the ordinary treatise now available.

Russell's Text-book on Hydraulics.

By GEORGE E. RUSSELL, Assistant Professor of Civil Engineering, Massachusetts Institute of Technology. viii + 183 pp. 8vo. $2.50.

This book is designed primarily for classroom use rather than for reference for practicing engineers. It avoids discussion of specialized topics which are taught separately with special books and devotes itself to the consideration of the more common and important subjects. At the end of each chapter are given problems to illustrate the application of the principles just preceding.

Benjamin's Machine Design.

By CHARLES H. BENJAMIN, Professor in Purdue University. 12mo. 202 pp. $2.00.

MACHINERY:—We know of no work on Machine-Design which can be more heartily recommended to the average student than this. . . . The work has the characteristics of Professor Benjamin's writing in general; that is, clearness and simplicity. It is brought up to date, containing, for example, a summary of the paper on the collapsing strength of lap-welded steel tubes presented by Professor Stewart before the spring meeting of the A. S. M. E. in 1906.

Leffler's The Elastic Arch.

With special reference to the Reinforced Concrete Arch. By BURTON R. LEFFLER, Engineer of Bridges on the Lake Shore and Michigan Southern Railway. viii + 57 pp. 12mo. $1.00.

HENRY HOLT AND COMPANY

NEW YORK CHICAGO

MATHEMATICAL SERIES

While this series has been planned to meet the needs of the student who is preparing for engineering work, it is hoped that it will serve equally well the purposes of those schools where mathematics is taken as an element in a liberal education. In order that the applications introduced may be of such character as to interest the general student and to train the prospective engineer in the kind of work which he is most likely to meet, it has been the policy of the editors to select, as joint authors of each text, a mathematician and a trained engineer or physicist.

The problems as well as the applications introduced in the text are of such a character as to draw upon the student's general information which will be of use to him later in the application of mathematics. Without sacrificing the value of mathematical study as a discipline, it is the purpose of the series so to correlate the mathematics with the physical applications as to stimulate the interest and train the student to use his mathematics as a means of investigation and stating the laws of physical phenomena.

The following texts are ready:

I. Calculus.

By E. J. TOWNSEND, Professor of Mathematics in the University of Illinois, and G. A. GOODENOUGH, Professor of Mechanical Engineering, University of Illinois. $2.50.

II. Essentials of Calculus.

By E. J. TOWNSEND and G. A. GOODENOUGH. $2.00.

III. College Algebra.

By H. L. RIETZ, Assistant Professor of Mathematics in the University of Illinois, and DR. A. R. CRATHORNE, Associate in Mathematics in the University of Illinois. $1.40.

IV. Plane Trigonometry, with Trigonometric and Logarithmic Tables.

By A. G. HALL, Professor of Mathematics in the University of Michigan, and F. H. FRINK, Professor of Railway Engineering in the University of Oregon. $1.25.

V. Plane and Spherical Trigonometry.

(Without Tables)

By A. G. HALL and F. H. FRINK. $1.00.

VI. Trigonometric and Logarithmic Tables.

By A. G. HALL and F. H. FRINK. 75 cents.

HENRY HOLT AND COMPANY

NEW YORK CHICAGO

E. A. NORDSTEDT

E. A. NORDSTEDT

89090523325 [redacted]

B89090523325A

SV
H79

KURT F. WENDT LIBRARY
COLLEGE OF ENGINEERING
UNIVERSITY OF WISCONSIN
MADISON, WI 53706